

GENERALIZED DIFFERENCE METHODS FOR DIFFERENTIAL EQUATIONS

Numerical Analysis of
Finite Volume Methods

Ronghua Li
Zhongying Chen
Wei Wu

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**Numerical Analysis of
Finite Volume Methods**

Ronghua Li

*University of Jilin
Changchun, China*

Zhongying Chen

*Zhongshan University
Guangzhou, China*

Wei Wu

*Dalian University of Technology
Dalian, China*



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Finite Volume Methods**

Ronghua Li

*University of Jilin
Changchun, China*

Zhongying Chen

*Zhongshan University
Guangzhou, China*

Wei Wu

*Dalian University of Technology
Dalian, China*



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Preface

Finite difference (on rectangular networks) and finite element methods are the two most important classes of numerical methods for partial differential equations. The finite difference method is particularly preferred for hyperbolic equations, especially quasi-linear ones which admit discontinuous solutions. The main defects of the difference method are: the considerable geometrical error of the approximation of curved domains by rectangular grids; the lack of a united and effective approach to deal with natural and internal boundary conditions; the difficulty to construct difference schemes with high accuracy, unless we allow the difference equation to relate more nodal points (which will in turn further increase the difficulty in dealing with boundary conditions). In 1953, R.H. MacNeal used integral interpolation (or integral balance) methods to establish difference schemes on irregular networks. These schemes reduce the geometrical error and, in particular, provide a united and effective approach to handle natural and internal boundary conditions, marking a significant advance in the development of difference methods. But in the following two decades, MacNeal's method did not attract much attention, perhaps because people had turned their attention to finite element methods. Among the few people doing research in the field at that time, I would like to mention A.M. Winslow (1967) and the engineering and mechanics group at Dalian Institute of Technology (1973). They employed the linear finite elements to construct difference schemes on arbitrary triangulations (besides the circumcenter dual grid discussed by MacNeal, they also considered the barycenter dual grid), and applied them to the computation of the electromag-

netic fields and the stress of elastic bodies. Since the late seventies, there have been series of papers on difference methods on irregular networks in the former Soviet Union and the former East Germany. Basically they followed MacNeal's approach to construct difference schemes, and adopted the framework of classical difference methods to establish *a priori* estimates (especially the extremum principle), convergences and error estimates. These results were included in B. Heinrich's monograph: Finite Difference Methods on Irregular Networks, ISNM 82, 1987.

Although the difference method on irregular networks has successfully reduced the geometrical error and overcome the difficulty in dealing with natural boundary conditions, it has not resolved the problem of constructing high accuracy difference schemes, and hence cannot match finite element methods in this respect. Besides, its error estimates require too many restrictions and are usually not optimal. Therefore, the theory of finite difference methods is still not as perfect as that of finite element methods. In 1978, the first author of this book utilized finite element spaces and generalized characteristic functions on dual elements, i.e., the common terms of the local Taylor expansions, to rewrite integral interpolation methods in a form of generalized Galerkin methods, and thus obtained a generalization of difference methods on irregular networks, that is, the so-called *generalized difference methods* (GDM for short). Since then, extensive research has been carried out on the theory and application of GDM, such as constructing linear and high order difference schemes for elliptic, parabolic and hyperbolic equations, establishing optimal error estimates in Sobolev norms, and applying GDM to underground fluids, electromagnetic fields and other practical problems. Both the theoretical observations and the computational experiments show that GDM enjoy not only the simplicity of difference methods but also the accuracy of finite element methods. To elaborate, the advantages of GDM are summarized as follows:

1. The grid is flexible (allowing, e.g., triangular and quadrilateral grids), the geometrical error is small, and the natural boundary conditions are easy to deal with.
2. The computational effort is greater than in (classical) finite

difference methods and less than finite element methods, while the accuracy is higher than with finite difference methods and nearly the same as with finite element methods. (The orders of the error estimates of GDM are the same as those of finite element methods, but practical computations show that finite element methods perform slightly better than GDM, perhaps due to the different magnitude of the constants in the error estimates.)

3. The mass conservation law is maintained, which is fairly desirable for, e.g., fluid and underground fluid computations.

4. The theory of GDM is almost as perfect as that of finite element methods. On the other hand, a special case of first order GDM leads to a general and united theory for difference methods on regular and irregular networks.

5. The variational form of GDM (the generalized Galerkin form) is helpful to connect the theories and algorithms of finite element and finite difference methods.

Therefore, GDM are meaningful generalization of difference methods and their further development seems promising. In 1994, the first two authors of this work wrote a book *Generalized Difference Methods for Differential Equations*, published (in Chinese) by Jilin University Press, in which is summarized the research of Chinese researchers on this topic in the preceding ten years.

At the end of the seventies and the beginning of the eighties, some computational fluid researchers (e.g., S. V. Patankar and A. Jameson among others) proposed to apply the difference method on irregular networks to the computation of compressible and incompressible fluid equations. Due to its many advantages, in particular its inheritance of the mass conservation law, this method developed rapidly, and by the end of the seventies, it had become one of the most efficient methods for fluid computation. This method appeared under many different names in the literature. But by the end of the eighties, people usually called it the finite volume method, or finite control volume method, indicating that it is a discrete approximation of the control equation in an integral form. This method is basically equivalent to the generalized difference method with piecewise constant and piecewise linear elements. Using the finite volume method to

construct numerical schemes for nonlinear conservative equations actually amounts to generalizing the classical difference schemes (such as Godunov or TVD schemes) to arbitrary grids (including triangular and tetrahedral grids). Not until the end of the eighties, did the numerical analysts get involved in the research of the finite volume method, and by now they have taken it as one of their favorite topics. For the convenience of international communication, we rewrote the Chinese edition of this book in English, and supplemented it with some new materials and recent important references. The original name *Generalized Difference Methods for Differential Equations* survives, but we have added the subtitle *Numerical Analysis of Finite Volume Methods*. In this way we expect to indicate that, on one hand, the generalized difference method is an extension and a development of the finite volume method, and on the other hand, it also provides from another angle a theoretical basis for the finite volume method.

This book is divided into eight chapters and arranged as follows.

Some preliminary materials are gathered in Chapter 1, such as a discussion of Sobolev spaces and the basic results of variational problems and their approximations. In particular, an abstract framework of the generalized difference method is provided in this chapter for later use.

Chapters 2 (except §7) and 3 discuss GDM for one- and two-dimensional second order linear elliptic equations, construct the generalized difference schemes with first-, second- and third-order elements, and establish some fairly comprehensive H^1 and L^2 error estimates, including certain superconvergence estimates. These results are basically parallel to those of finite element methods, but usually more difficult to prove.

GDM are extended to second-order nonlinear elliptic equations and biharmonic equations in Chapter 4 (and §7 of Chapter 2). As the orders of the partial differential equations increase, the nonconforming feature of GDM becomes more evident, making it more difficult to construct schemes and estimate errors. We introduce in this chapter the GDM based on mixed variational principles (§§2-3) and certain modified variational principles (§§3-4). The corresponding error estimates are also presented.

The GDM for parabolic equations are treated in Chapter 5 in a way similar to the corresponding finite element methods.

The GDM for hyperbolic equations, especially the first order systems, are considered in Chapter 6. The GDM for elliptic equations cannot be directly extended here. Instead we modify a discontinuous finite element method to obtain a generalized upwind scheme with high accuracy. The convergence order is shown. §4 of this chapter discusses briefly the finite volume method for nonlinear conservative equations, and the corresponding references are provided.

Chapter 7 presents the GDM for convection-dominated diffusion equations. The basic idea is to use GDM to discretize the diffusion term, and upwind or high accuracy upwind schemes to the convection term. For the sake of comparison, we also outline in §1 of this chapter the characteristic difference method proposed by Douglas and Russel in the early eighties.

Chapter 8 is devoted to the applications of GDM to plane elasticity problems, electromagnetic fields, groundwater contaminations, Stokes equations, coupled sound-heat flows, and the regularized long wave equations. By virtue of the variational form of GDM, we are also able to extend the hierarchical basis methods for finite element equations to difference equations.

A *Bibliography and Comments* section is attached to the end of each chapter. A complete (to the best of our knowledge) bibliography is provided at the end of the book, which is divided into three groups: A in Chinese, B in English (including a few papers in German and French), and C in Russian.

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Ronghua Li

Contents

PREFACE	iii
1 PRELIMINARIES	1
1.1 Sobolev Spaces	1
1.1.1 Smooth approximations. Fundamental lemma of variational methods	1
1.1.2 Generalized derivatives and Sobolev spaces . .	3
1.1.3 Imbedding and trace theorems	7
1.1.4 Finite element spaces	9
1.1.5 Interpolation error estimates in Sobolev spaces	15
1.2 Variational Problems and Their Approximations . . .	19
1.2.1 Abstract variational form	19
1.2.2 Green's formulas and variational problems . . .	23
1.2.3 Well-posedness of variational problems	27
1.2.4 Approximation methods. A necessary and suf- ficient condition for approximate-solvability . .	29
1.2.5 Galerkin methods	35
1.2.6 Generalized Galerkin methods	38
Bibliography and Comments	43
2 TWO POINT BOUNDARY VALUE PROBLEMS	45
2.1 Basic Ideas of the Generalized Difference Method	45
2.1.1 A variational form	45
2.1.2 Galerkin methods	47
2.1.3 Generalized Galerkin variational principles . .	48

2.1.4	Generalized difference methods	52
2.2	Linear Element Difference Schemes	53
2.2.1	Trial and test function spaces	53
2.2.2	Difference equations	56
2.2.3	Convergence estimates	57
2.3	Quadratic Element Difference Schemes	61
2.3.1	Trial and test spaces	61
2.3.2	Difference equations	63
2.3.3	Convergence order estimates	66
2.4	Cubic Element Difference Schemes	72
2.4.1	Trial and test spaces	73
2.4.2	Generalized difference methods	76
2.4.3	Some lemmas	79
2.4.4	Existence, uniqueness and stability	84
2.4.5	Convergence order estimates	86
2.4.6	Numerical examples	87
2.5	Estimates in L^2 and Maximum Norms	88
2.5.1	L^2 -estimates	88
2.5.2	Maximum norm estimates	91
2.6	Superconvergence	92
2.6.1	Optimal stress points	93
2.6.2	Superconvergence for linear element difference schemes	96
2.6.3	Superconvergence for cubic element difference schemes	97
2.7	Generalized Difference Methods for a Fourth Order Equation	101
2.7.1	Generalized difference equations	101
2.7.2	Positive definiteness of $a(u_h, \Pi_h^* u_h)$	104
2.7.3	Convergence order estimates	106
2.7.4	Numerical examples	108
	Bibliography and Comments	108
3	SECOND ORDER ELLIPTIC EQUATIONS	111
3.1	Introduction	111
3.2	Generalized Difference Methods on Triangular Meshes	114

3.2.1	Trial and test function spaces	114
3.2.2	Generalized difference equation	119
3.2.3	<i>a priori</i> estimates	124
3.2.4	Error estimates	129
3.3	Generalized Difference Methods on Quadrilateral Meshes	131
3.3.1	Trial and test function spaces	131
3.3.2	Generalized difference equation	134
3.3.3	Convergence order estimates	137
3.4	Quadratic Element Difference Schemes	139
3.4.1	Trial and test function spaces	139
3.4.2	Generalized difference equation	141
3.4.3	<i>a priori</i> estimates	145
3.4.4	Error estimates	150
3.4.5	Numerical example	152
3.5	Cubic Element Difference Schemes	154
3.5.1	Trial and test function spaces	154
3.5.2	Generalized difference equations	156
3.5.3	<i>a priori</i> estimates	158
3.5.4	Error estimates	164
3.6	L^2 and Maximum Norm Estimates	167
3.6.1	L^2 estimates	167
3.6.2	A maximum estimate and some remarks	173
3.7	Superconvergences	174
3.7.1	Weak estimate of interpolations	174
3.7.2	Superconvergence estimates	181
	Bibliography and Comments	182
4	FOURTH ORDER AND NONLINEAR ELLIPTIC EQUATIONS	187
4.1	Mixed Generalized Difference Methods Based on Ciarlet-Raviart Variational Principle	187
4.1.1	Mixed generalized difference equations	188
4.1.2	Error estimates	193

4.2	Mixed Generalized Difference Methods Based on Hermann-Miyoshi Variational Principle	196
4.2.1	Mixed generalized difference equations	197
4.2.2	Numerical experiments	198
4.3	Nonconforming Generalized Difference Method Based on Zienkiewicz Elements	202
4.3.1	Variational principle	202
4.3.2	Generalized difference schemes based on Zienkiewicz elements	204
4.3.3	Error analyses	207
4.3.4	Numerical experiment	218
4.4	Nonconforming Generalized Difference Methods Based on Adini Elements	220
4.4.1	Generalized difference scheme	220
4.4.2	Error estimate	222
4.4.3	Numerical example	226
4.5	Second Order Nonlinear Elliptic Equations	227
4.5.1	Generalized difference scheme	227
4.5.2	Error estimate	231
	Bibliography and Comments	233
5	PARABOLIC EQUATIONS	235
5.1	Semi-discrete Generalized Difference Schemes	235
5.1.1	Problem and schemes	235
5.1.2	Some lemmas	238
5.1.3	L^2 -error estimate	241
5.1.4	H^1 -error estimate	242
5.2	Fully-discrete Generalized Difference Schemes	244
5.2.1	Fully-discrete schemes	244
5.2.2	Error estimates for backward Euler generalized difference schemes	245

5.2.3	Error estimates for Crank-Nicolson generalized difference schemes	250
5.3	Mass Concentration Methods	254
5.3.1	Construction of schemes	254
5.3.2	Error estimates for semi-discrete schemes	256
5.3.3	Error estimates for fully-discrete schemes	258
5.4	High Order Element Difference Schemes	261
5.4.1	Cubic element difference schemes for one-dimensional parabolic equations	261
5.4.2	Quadratic element difference schemes for two-dimensional parabolic equations	267
5.5	Generalized Difference Methods for Nonlinear Parabolic Equations	272
5.5.1	Problem and schemes	272
5.5.2	Some lemmas	276
5.5.3	Error estimates	283
	Bibliography and Comments	285
6	HYPERBOLIC EQUATIONS	287
6.1	Generalized Difference Methods for Second Order Hyperbolic Equations	287
6.1.1	Semi-discrete generalized difference scheme	288
6.1.2	Fully-discrete generalized difference scheme	292
6.2	Generalized Upwind Schemes for First Order Hyperbolic Equations	296
6.2.1	Generalized upwind schemes	297
6.2.2	Semi-discrete error estimates	300
6.2.3	Fully-discrete error estimates	303
6.3	Generalized Upwind Schemes for First Order Hyperbolic Systems	307
6.3.1	Integral forms	307
6.3.2	Generalized upwind difference schemes	309
6.3.3	Estimation of a bilinear form	310
6.3.4	Some practical difference schemes	313
6.3.5	A numerical example	316

6.4	Finite Volume Methods for Nonlinear Conservative Hyperbolic Equations	317
	Bibliography and Comments	323
7	CONVECTION-DOMINATED DIFFUSION PROBLEMS	325
7.1	One-Dimensional Characteristic Difference Schemes	326
7.1.1	Difference methods based on algebraic interpo- lations	328
7.1.2	Upwind difference schemes	330
7.2	Generalized Upwind Difference Schemes for Steady-state Problems	332
7.2.1	Construction of the difference schemes	333
7.2.2	Convergence and error estimate	336
7.2.3	Extreme value theorem and uniform convergence	339
7.2.4	Mass conservation	344
7.3	Generalized Upwind Difference Schemes for Nonsteady-state Problems	345
7.3.1	Construction of difference schemes	346
7.3.2	Convergence and error estimate	348
7.4	Highly Accurate Generalized Upwind Schemes	351
7.4.1	Construction of the difference schemes	351
7.4.2	Convergence and error estimate	355
7.5	Upwind Schemes for Nonlinear Convection Problems	357
	Bibliography and Comments	360
8	APPLICATIONS	361
8.1	Planar Elastic Problems	361
8.1.1	Displacement methods	362
8.1.2	Mixed methods	365
8.2	Computation of Electromagnetic Fields	367
8.3	Numerical Simulation of Underground Water Pollution	373
8.3.1	Generalized difference scheme	374
8.3.2	Generalized upwind difference schemes	378

8.3.3	Upwind weighted multi-element balancing method	382
8.4	Stokes Equation	385
8.4.1	Nonconforming generalized difference method	386
8.4.2	Convergence and error estimate	389
8.4.3	A numerical example	393
8.5	Coupled Sound-Heat Problems	394
8.6	Regularized Long Wave Equations	399
8.6.1	Semi-discrete generalized difference schemes	399
8.6.2	Fully-discrete generalized difference schemes	404
8.6.3	Numerical experiments	407
8.7	Hierarchical Basis Methods	409
8.7.1	Hierarchical Basis	410
8.7.2	Application to difference equations	413
8.7.3	Iteration methods	416
8.7.4	Numerical experiments	417
	Bibliography and Comments	418
	Bibliography	421
	Index	439

Chapter 1

PRELIMINARIES

1.1 Sobolev Spaces

Sobolev spaces and their interpolations are basic tools for numerical solutions of partial differential equations. The main related results are outlined in this section, cf. [A-19] and [B-1] for details.

1.1.1 Smooth approximations. Fundamental lemma of variational methods

Let R^n be an n -dimensional Euclidean space and Ω a region in R^n . $L^p(\Omega)$ ($1 \leq p < \infty$) denotes the set of all the functions defined on Ω of which the p -th powers are integrable, and $L^\infty(\Omega)$ all essentially bounded (i.e. bounded except on a zero measure set) measurable functions. $L^p(\Omega)$ becomes a Banach space if supplied with a norm

$$\|u\|_p = \begin{cases} \left(\int_{\Omega} |u(x)|^p dx \right)^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup}_{x \in \Omega} |u(x)| \equiv \inf_{me=0} \sup_{x \in \Omega-e} |u(x)|, & p = \infty. \end{cases}$$

Here me denotes the Lebesgue measure of the set e . Denote by $C^m(\Omega)$ the set of m -th continuously differentiable functions defined on Ω , and $C^\infty(\Omega)$ of infinitely differentiable functions. $C^0(\Omega)$ is simplified as $C(\Omega)$. The closure of the set $\{x \in \Omega : u(x) \neq 0\}$ is called the support of the function u and denoted by $\text{supp } u$. $C_0^m(\Omega)$ and $C_0^\infty(\Omega)$

are subsets of $C^m(\Omega)$ and $C^\infty(\Omega)$, respectively, containing functions with compact supports in Ω .

Take any function $j(x)$ satisfying the following conditions:

- (i) $j(x) \in C_0^\infty(R^n)$;
- (ii) $j(x) \geq 0$, $j(x) = 0$ when $|x| > 1$;
- (iii) $\int_{R^n} j(x) dx = 1$.

For example, we can set

$$j(x) = \begin{cases} \frac{1}{\gamma} \exp\left(-\frac{|x|^2}{1-|x|^2}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where

$$\gamma = \int_{|x| < 1} \exp\left(-\frac{|x|^2}{1-|x|^2}\right) dx.$$

Definition 1.1.1 *An integral operator J_ϵ*

$$J_\epsilon u(x) = \int_{R^n} j_\epsilon(x, y) u(y) dy$$

with a kernel

$$j_\epsilon(x, y) = \frac{1}{\epsilon^n} j\left(\frac{x-y}{\epsilon}\right) \quad (\epsilon > 0)$$

is called a smoothing operator, and $J_\epsilon u$ an averaging function.

The following theorem summarizes the main properties of the averaging function.

Theorem 1.1.1 (Average approximation theorem) *For any function $u \in L^p(\Omega)$ ($1 \leq p < \infty$), we define its value to be zero outside of Ω . Then we have*

- (i) $J_\epsilon u \in C^\infty(R^n) \cap L^p(\Omega)$, and $J_\epsilon u \in C_0^\infty(R^n)$ when $\text{supp } u$ is bounded;
- (ii) $\|J_\epsilon u\|_p \leq \|u\|_p$;
- (iii) $\lim_{\epsilon \rightarrow 0} \|J_\epsilon u - u\|_p = 0$.

Theorem 1.1.1 indicates that the functions in $L^p(\Omega)$ ($1 \leq p < \infty$) can be approximated by sufficiently smooth functions. In other words, $C^\infty(\bar{R}^n)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$. Furthermore, we have the following theorem.

Theorem 1.1.2 *If $1 \leq p < \infty$, then $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$.*

The following theorem can be proved by the average approximation theorem.

Theorem 1.1.3 (Fundamental lemma of variational methods) *If $u \in L^p(\Omega)$ ($1 \leq p < \infty$) satisfies*

$$\int_{\Omega} u \phi dx = 0, \quad \forall \phi \in C_0^\infty(\Omega),$$

then $u = 0$ almost everywhere on Ω .

Proof In fact, $j_\epsilon(x, y) \in C_0^\infty(\Omega)$ for any $0 < \epsilon < \delta$ and $x \in \Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$. So we have

$$J_\epsilon u(x) = \int_{\Omega} j_\epsilon(x, y) u(y) dy = 0, \quad \forall x \in \Omega_\delta.$$

It follows from Theorem 1.1.1 that

$$\|u\|_{L^p(\Omega_\delta)} = \|u - J_\epsilon u\|_{L^p(\Omega_\delta)} \leq \|u - J_\epsilon u\|_{L^p(\Omega)} \rightarrow 0 \quad (\text{as } \epsilon \rightarrow 0).$$

This leads to the desired conclusion. $\square$

1.1.2 Generalized derivatives and Sobolev spaces

Write the partial derivative of the function u as

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -index, $\alpha_1, \dots, \alpha_n$ are non-negative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Definition 1.1.2 Let $L^1_{\text{loc}}(\Omega)$ be the local Lebesgue integrable function space and $u \in L^1_{\text{loc}}(\Omega)$. If there exists a $v \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_{\Omega} v \phi dx = (-1)^{|\alpha|} \int_{\Omega} u D^{\alpha} \phi dx, \quad \forall \phi \in C_0^{\infty}(\Omega),$$

then we call v an $|\alpha|$ -th generalized derivative of u and write $v = D^{\alpha}u$.

By the fundamental lemma of variational methods, a generalized derivative must be unique as long as it exists. It is easy to show that if a classical derivative of u exists and belongs to $L^2(\Omega)$, then its generalized derivative also exists and is identical with the classical derivative. Hence the generalized derivative is indeed a generalization of the classical one.

Generalized derivatives enjoy the following properties:

- (i) $D^{\alpha}(au + bv) = aD^{\alpha}u + bD^{\alpha}v$ (a, b are constants),
- (ii) $D^{\alpha+\beta}u = D^{\alpha}(D^{\beta}u)$,
- (iii) $D(uv) = vDu + uDv$ ($D = \frac{\partial}{\partial x_k}$),
- (iv) $D^{\alpha}u = 0$ for all α with $|\alpha| = m$, if and only if u equals to an $(m-1)$ -th polynomial almost everywhere.

Definition 1.1.3 Let m be a non-negative integer and $1 \leq p \leq \infty$. Set

$$W^{m,p}(\Omega) \equiv \{u \in L^p(\Omega) : D^{\alpha}u \in L^p(\Omega), \forall \alpha, 0 \leq |\alpha| \leq m\},$$

and supply it with a norm $\|\cdot\|_{m,p}$

$$\|u\|_{m,p} = \begin{cases} \left(\sum_{0 \leq |\alpha| \leq m} \|D^{\alpha}u\|_p^p \right)^{1/p}, & \text{for } 1 \leq p < \infty, \\ \max_{0 \leq |\alpha| \leq m} \|D^{\alpha}u\|_{\infty}, & \text{for } p = \infty. \end{cases}$$

Define $W_0^{m,p}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ with respect to the norm $\|\cdot\|_{m,p}$. The normed linear spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are called Sobolev spaces on Ω .

In particular when $p = 2$ we write $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ as $H^m(\Omega)$ and $H_0^m(\Omega)$, respectively. It is an easy matter to see that $W^{0,p}(\Omega) = L^p(\Omega)$ and $H^0(\Omega) = L^2(\Omega)$.

$W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are obviously Banach spaces, and $H^m(\Omega)$ and $H_0^m(\Omega)$ are Hilbert spaces equipped with an inner product

$$(u, v)_m = \sum_{0 \leq |\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}, \quad u, v \in H^m(\Omega).$$

The norm $\|\cdot\|_{m,p}$ is written as $\|\cdot\|_{m,p,\Omega}$ when the region needs to be specified, and as $\|\cdot\|_m$ when $p = 2$ and there is no danger of confusion. We can also introduce a $|\cdot|_{m,p}$ semi-norms

$$|u|_{m,p} = \begin{cases} \left(\sum_{|\alpha|=m} \|D^\alpha u\|_{0,p}^p \right)^{1/p}, & \text{for } 1 \leq p < \infty, \\ \max_{|\alpha|=m} \|D^\alpha u\|_{0,\infty}, & \text{for } p = \infty. \end{cases}$$

The following theorem on equivalent norms can be proved by the compact imbedding theorem given later on.

Theorem 1.1.4 (Equivalent norm theorem) *Suppose $\Omega \subset R^n$ is a bounded L -region; $m \geq 1$; $1 \leq p \leq \infty$; and $l_1, \dots, l_N$ are bounded linear functionals on $W^{m,p}(\Omega)$ and they are not simultaneously equal to zero on any nonzero polynomial of degree less than or equal to $m - 1$. Then the functional*

$$\|u\| \equiv |u|_{m,p} + \sum_{j=1}^N |l_j(u)|$$

on $W^{m,p}(\Omega)$ is an equivalent norm, that is, there exist constants $\alpha, \beta > 0$ such that

$$\alpha \|u\| \leq |u|_{m,p} \leq \beta \|u\|, \quad \forall u \in W^{m,p}(\Omega).$$

Remark By an L -region Ω we mean that Ω has a local Lipschitz boundary, that is, there is a neighborhood U_x for each point x on the boundary of Ω such that $\partial\Omega \cap U_x$ can be expressed as a Lipschitz continuous function with respect to certain local Cartesian coordinates.

By virtue of the above theorem and the trace theorem (Theorem 1.1.9 below) we know that $\|u\| = |u|_{1,p} + \left| \int_{\partial\Omega} u ds \right|$ is an equivalent norm for $W^{1,p}(\Omega)$. So there exists $\beta > 0$ such that

$$|u|_{0,p} \leq \beta |u|_{1,p}, \quad \forall u \in W_0^{1,p}(\Omega).$$

Using this and the inductive method leads to the following theorem.

Theorem 1.1.5 *Let $\Omega \in R^n$ be a bounded L -region and $m \geq 0$, $1 \leq p < \infty$, then $|u|_{m,p}$ is an equivalent norm for $W_0^{m,p}(\Omega)$.*

Some important properties of Sobolev spaces are given in the following theorem.

Theorem 1.1.6 *Let $\Omega \in R^n$ be a region and $m \geq 1$. Then we have the following:*

- (i) $W^{m,p}(\Omega)$ ($1 \leq p < \infty$) is separable.
- (ii) $W^{m,p}(\Omega)$ ($1 < p < \infty$) is reflexive and uniformly convex.
- (iii) $\{u \in C^\infty(\Omega) : \|u\|_{m,p} < \infty\}$ is dense in $W^{m,p}(\Omega)$ ($1 \leq p < \infty$); so $C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$ ($1 \leq p < \infty$); and $C^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$ ($1 \leq p < \infty$) when Ω is a bounded L -region.

Property (iii) enables us to make an equivalent definition when Ω is an L -region:

$W^{m,p}(\Omega) \equiv$ the completion of $C^\infty(\overline{\Omega})$ under the norm $\|\cdot\|_{m,p}$.

Next we introduce the Sobolev spaces with negative index. For $1 < p < \infty$, let $p' = p/(p-1)$ be its conjugate index. Write

$$\langle u, v \rangle = \int_{\Omega} u(x)v(x)dx.$$

For any $v \in L^{p'}(\Omega)$ we define a bounded linear functional L_v on $W_0^{m,p}(\Omega)$

$$L_v(u) = \langle u, v \rangle, \quad \forall u \in W_0^{m,p}(\Omega),$$

and a corresponding norm

$$\|L_v\| = \sup_{u \in W_0^{m,p}(\Omega), \|u\|_{m,p} \leq 1} |L_v(u)|.$$

It can be verified that $V = \{L_v : v \in L^{p'}(\Omega)\}$ is dense in $(W_0^{m,p}(\Omega))'$, and hence its closure $\overline{V} = (W_0^{m,p}(\Omega))'$. (The notation B' denotes the dual space of a Banach space B .)

Definition 1.1.4 Let $1 < p < \infty$, $p' = p/(p-1)$, $v \in L^{p'}(\Omega)$. Define a negative norm of v by $\|v\|_{-m,p'}$

$$\|v\|_{-m,p'} \equiv \sup_{u \in W_0^{m,p}(\Omega), \|u\|_{m,p} \leq 1} |\langle u, v \rangle|.$$

Correspondingly we define the Sobolev spaces with negative index:

$W^{-m,p'}(\Omega) \equiv$ the completion of $L^{p'}(\Omega)$ in the norm $\|\cdot\|_{-m,p'}$.

When $p' = 2$ we write $H^{-m}(\Omega) = W^{-m,p'}(\Omega)$.

Notice that $L^{p'}(\Omega)$ and V are isometrically isomorphic, so we have the identification

$$W^{-m,p'}(\Omega) = (W_0^{m,p}(\Omega))'.$$

1.1.3 Imbedding and trace theorems

The next two theorems reveal some more profound properties of Sobolev spaces.

Definition 1.1.5 Let X and Y be two normed linear spaces. We say that X is imbedded in Y , written as $X \rightarrow Y$, if

- (i) $X \subset Y$,
- (ii) The identification operator I mapping $x \in X$ to $Ix \in Y$ is continuous, i.e., there exists a constant $M > 0$ such that

$$\|Ix\|_Y \leq M\|x\|_X, \quad \forall x \in X.$$

I is called an imbedding operator and M an imbedding constant.

Theorem 1.1.7 (Sobolev imbedding theorem) Suppose that $\Omega \subset R^n$ is a bounded L -region, that m, k are nonnegative constants and that $1 \leq p < \infty$; then

$$W^{m+k,p}(\Omega) \rightarrow W^{k,q}(\Omega) \text{ for } 1 \leq q \leq np/(n-mp) \text{ and } m < n/p;$$

$$W^{m+k,p}(\Omega) \rightarrow W^{k,q}(\Omega) \text{ for } 1 \leq q < \infty \text{ and } m = n/p;$$

$$W^{m+k,p}(\Omega) \rightarrow C^k(\bar{\Omega}) \text{ for } m > n/p.$$

In particular,

$$W^{m,p}(\Omega) \rightarrow L^q(\Omega) \text{ for } 1 \leq q \leq np/(n-mp) \text{ and } m < n/p;$$

$$W^{m,p}(\Omega) \rightarrow L^q(\Omega) \text{ for } 1 \leq q < \infty \text{ and } m = n/p;$$

$$W^{m,p}(\Omega) \rightarrow C(\bar{\Omega}) \text{ for } m > n/p.$$

Theorem 1.1.8 (Compact imbedding theorem) *Under the assumption of Theorem 1.1.7, the following imbedding operators are compact:*

$$\begin{aligned} W^{m+k,p}(\Omega) &\rightarrow W^{k,q}(\Omega) \text{ for } 1 \leq q < np/(n-mp) \text{ and } m < n/p; \\ W^{m+k,p}(\Omega) &\rightarrow W^{k,q}(\Omega) \text{ for } 1 \leq q < \infty \text{ and } m = n/p; \\ W^{m+k,p}(\Omega) &\rightarrow C^k(\bar{\Omega}) \text{ for } m > n/p. \end{aligned}$$

It should be pointed out that the elements of $W^{m,p}(\Omega)$ are in fact equivalent classes. Almost equal functions are said to be equivalent and classified into an equivalent class. $W^{m,p}(\Omega) \rightarrow C(\bar{\Omega})$ means that any $u \in W^{m,p}(\Omega)$ must be equivalent to a function in $C(\bar{\Omega})$, i.e., the equivalent class $u \in W^{m,p}(\Omega)$ contains an element belonging to $C(\bar{\Omega})$, and that there exists a constant M such that

$$\|u\|_{C(\bar{\Omega})} \leq M\|u\|_{m,p,\Omega}, \quad \forall u \in W^{m,p}(\Omega).$$

Now let us consider the boundary value of the functions of $H^m(\Omega)$, i.e., the trace $u|_{\partial\Omega}$ of u . Suppose a bounded region Ω possesses an m -th smooth boundary $\partial\Omega$. Since $\partial\Omega$ has zero measure in R^n , it is meaningless to talk in the usual sense about the value of u on the boundary $\partial\Omega$. Some precise and reasonable definition must be introduced. The idea is to employ the density of $C^m(\bar{\Omega})$ in $H^m(\Omega)$ to generalize the definition.

Definition 1.1.6 *Assume that $\Omega \subset R^n$ is a bounded region with an m -th smooth boundary $\partial\Omega$ and that $u \in C^m(\bar{\Omega})$. The linear operator $(\gamma_0, \gamma_2, \dots, \gamma_{m-1})$ is called a trace operator, where*

$$\gamma_j u = \frac{\partial^j u}{\partial n^j} \Big|_{\partial\Omega}, \quad 0 \leq j \leq m-1,$$

and $\frac{\partial^j}{\partial n^j}$ denotes the j -th directional derivative on the outer normal direction of $\partial\Omega$.

Lemma 1.1.1 *For the above mentioned region we have a constant $C > 0$ such that*

$$\|\gamma_j u\|_{0,\partial\Omega} \leq C\|u\|_{j+1,\Omega}, \quad \forall u \in C^m(\bar{\Omega}), \quad 0 \leq j \leq m-1.$$

For $u \in H^m(\Omega)$ we can choose a sequence $\{u_k\} \subset C^m(\bar{\Omega})$ such that $\|u - u_k\|_{m,\Omega} \rightarrow 0$ as $k \rightarrow \infty$. It follows from the lemma that $\{\gamma_j u_k\}$ is a Cauchy sequence in $L^2(\partial\Omega)$. So there is a limit $v_j \in L^2(\partial\Omega)$. v_j is obviously independent of the choice of $\{u_k\}$. Thus, we can define the trace of $u \in H^m(\Omega)$ on $\partial\Omega$ as

$$\gamma_j u = v_j = \lim_{k \rightarrow \infty} \gamma_j u_k.$$

Theorem 1.1.9 *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded region with an m -th smooth boundary and that $u \in H^m(\Omega)$. Then there exists a constant $C > 0$ independent of u such that*

$$\|\gamma_j u\|_{0,\partial\Omega} \leq C \|u\|_{j+1,\Omega}, \quad \forall 0 \leq j \leq m-1.$$

In particular,

$$\|u\|_{0,\partial\Omega} \leq C \|u\|_{1,\Omega}.$$

The last inequality (the imbedding $H^1(\Omega) \rightarrow L^2(\partial\Omega)$) only requires $\partial\Omega$ to be a Lipschitz continuous surface.

Finally we point out that it follows from the definition of the trace operator that

$$H_0^m(\Omega) = \left\{ u \in H^m(\Omega) : \gamma_j u = \frac{\partial^j u}{\partial n^j} \Big|_{\partial\Omega} = 0, \quad 0 \leq j \leq m-1 \right\}.$$

1.1.4 Finite element spaces

Essentially a numerical method for differential equations means discretizations of the infinite dimensional function spaces and approximations of the original equations by equations in finite dimensional spaces. The interpolation is a basic method to construct these finite dimensional spaces, and the error estimates of the approximate solutions and the true solutions often rely on the error estimates of the interpolate approximations. Our interpolate approximation problem involves Sobolev spaces, with the approximating finite dimensional spaces being piecewise polynomial spaces (finite element spaces) subject to certain constraints. The related concepts and results constitute the so-called interpolation theory of Sobolev spaces.

In this subsection, let us first introduce the finite element spaces. Let T_h be a decomposition of a region $\bar{\Omega}$, dividing $\bar{\Omega}$ into finite bounded closed sets K 's which possess Lipschitz continuous boundaries, share no common inner points and have nonempty interior. So $\bar{\Omega} = \bigcup_{K \in T_h} K$. Here K is called an element of T_h and h stands for the largest element diameter.

Definition 1.1.7 A finite dimensional space V_h is called a finite element space with respect to the decomposition T_h if we have the following:

(i) For each $K \in T_h$, the set $P_K \equiv \{p : p = v_h|_K, \forall v_h \in V_h\}$ is a family of polynomials. And there exists a set of freedoms $\Sigma_K = \{l_i, 1 \leq i \leq N\}$ (namely a set of linearly independent linear functionals, often presented as a group of parameters $\{\alpha_i, 1 \leq i \leq N\}$, c.f. the examples below), which is P_K -uniquely solvable: for any given $\{\alpha_i, 1 \leq i \leq N\}$ there exists a unique function $p \in P_K$ satisfying

$$l_i(p) = \alpha_i, \quad 1 \leq i \leq N;$$

(ii) The functions of V_h possess certain smoothness on Ω , e.g., $V_h \subset C^m(\bar{\Omega})$ (m is a non-negative integer).

The triple $\{K, P_K, \Sigma_K\}$ specifies a finite element space.

We observe the following fact:

$$V_h = \{v_h \in C^m(\bar{\Omega}) : v_h|_K \in P_K, K \in T_h\} \subset H^{m+1}(\Omega).$$

Next we give some examples of finite elements.

Triangulation. Suppose $\bar{\Omega} \subset R^2$ can be decomposed into finite triangles such that different triangles have no overlap interior region, and a vertex of any triangle does not belong to the interior of a side of any other triangle. All such triangles form a decomposition of $\bar{\Omega}$, called a triangulation and denoted by $T_h = \{K\}$. The element K with vertexes a_1, a_2 and a_3 can be expressed as

$$K = \left\{ (x, y) : (x, y) = \sum_{i=1}^3 \lambda_i a_i, \quad 0 \leq \lambda_i \leq 1 \quad (1 \leq i \leq 3), \quad \sum_{i=1}^3 \lambda_i = 1 \right\}.$$

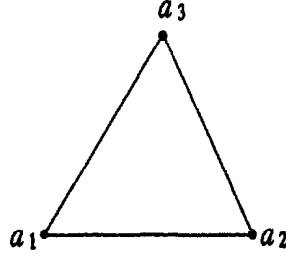


Fig. 1.1.1

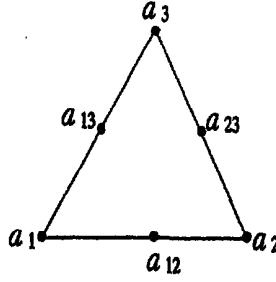


Fig. 1.1.2

$\lambda_i = \lambda_i(x, y)$ ($i = 1, 2, 3$) are called area coordinates of the point (x, y) . Denote by $\mathcal{P}_k(K)$ the set of polynomials on K of degrees less than or equal to k .

Example 1 Lagrange linear element.

$K = \triangle a_1 a_2 a_3$ (c.f. Fig.1.1.1);

$P_K = \mathcal{P}_1(K)$, $\dim P_K = 3$;

$\Sigma_K = \{p(a_i) : i = 1, 2, 3\}$.

Any $p \in P_K$ is determined uniquely by its values on the vertexes a_1, a_2 and a_3 :

$$p = \sum_{i=1}^3 p(a_i) \lambda_i.$$

It is easy to see that the corresponding finite element space $V_h \subset C(\bar{\Omega})$, and hence $V_h \subset H^1(\Omega)$.

Example 2 Lagrange quadratic element.

$K = \triangle a_1 a_2 a_3$, and the midpoints $a_{ij} = \frac{1}{2}(a_i + a_j)$ (see Fig.1.1.2);

$P_K = \mathcal{P}_2(K)$, $\dim P_K = 6$;

$\Sigma_K = \{p(a_i), 1 \leq i \leq 3; p(a_{ij}), 1 \leq i < j \leq 3\}$.

Any $p \in P_K$ is determined uniquely by its values on the vertexes and

the midpoints:

$$p = \sum_{i=1}^3 \lambda_i (2\lambda_i - 1) p(a_i) + \sum_{i < j} 4\lambda_i \lambda_j p(a_{ij}).$$

The corresponding finite element space $V_h \subset C(\bar{\Omega}) \cap H^1(\Omega)$.

Example 3 Hermite cubic element.

$K = \Delta a_1 a_2 a_3$, and the barycenter $a_{123} = \frac{1}{3}(a_1 + a_2 + a_3)$;

$P_K = \mathcal{P}_3(K)$, $\dim P_K = 10$;

$$\Sigma_K = \left\{ p(a_i), \frac{\partial p(a_i)}{\partial x}, \frac{\partial p(a_i)}{\partial y}, 1 \leq i \leq 3; p(a_{123}) \right\}.$$

Any $p \in P_K$ has the following expression:

$$p = \sum_{i=1}^3 (-2\lambda_i^3 + 3\lambda_i^2 - 7\lambda_1\lambda_2\lambda_3) p(a_i) + 27\lambda_1\lambda_2\lambda_3 p(a_{123}) \\ + \sum_{i \neq j} \lambda_i \lambda_j (2\lambda_i + \lambda_j - 1) Dp(a_i) \cdot (a_j - a_i),$$

where $Dp(a) = \left(\frac{\partial p(a)}{\partial x}, \frac{\partial p(a)}{\partial y} \right)$. The corresponding finite element space $V_h \subset C(\bar{\Omega}) \cap H^1(\Omega)$.

Example 4 Restricted Hermite cubic element (Zienkiewicz element).

$K = \Delta a_1 a_2 a_3$;

$$P_K = \left\{ p \in \mathcal{P}_3(K) : p(a_{123}) = \frac{1}{3} \sum_{i=1}^3 p(a_i) - \frac{1}{6} \sum_{i=1}^3 Dp(a_i) \cdot (a_i - a_{123}) \right\},$$

$\dim P_K = 9$, $P_K \supset \mathcal{P}_2(K)$;

$$\Sigma_K = \left\{ p(a_i), \frac{\partial p(a_i)}{\partial x}, \frac{\partial p(a_i)}{\partial y}, 1 \leq i \leq 3 \right\}.$$

The expression of $p \in P_K$ can be obtained by inserting the constraint condition on $p(a_{123})$ in the definition of P_K into the second term of

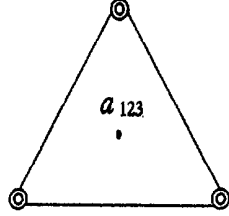


Fig. 1.1.3

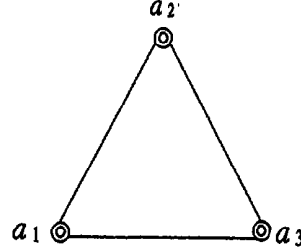


Fig. 1.1.4

the expression in Example 3. The corresponding finite element space $V_h \subset C(\bar{\Omega}) \cap H^1(\Omega)$.

Rectangular grid. Suppose the region $\bar{\Omega} \subset R^2$ can be divided into a sum of finite number of rectangles with each side of the rectangles being parallel to the axes of coordinates. Any two different rectangles either are disjoint or share a common side or vertex. All the rectangles constitute a grid of $\bar{\Omega}$, called a rectangular grid. The affine mapping,

$$\xi = (x - x_i)/\Delta x, \quad \eta = (y - y_i)/\Delta y$$

maps the rectangle

$$K = \{(x, y) : x_i \leq x \leq x_i + \Delta x, y_i \leq y \leq y_i + \Delta y\}$$

onto a unit square $[0, 1; 0, 1]$. Denote by $\mathcal{Q}_k$ the set of polynomials of x and y with degrees less than or equal to k . Note $\mathcal{P}_k \subset \mathcal{Q}_k \subset \mathcal{P}_{2k}$.

Example 5 Lagrange bilinear element (c.f. Fig. 1.1.5).

$$K = \text{rectangle } a_1 a_2 a_3 a_4;$$

$$P_K = \mathcal{Q}_1(K), \dim P_K = 4;$$

$$\Sigma_K = \{p(a_i), 1 \leq i \leq 4\};$$

$$\forall p \in P_K, \quad p = \sum_{i=1}^4 p(a_i) \mu_i,$$

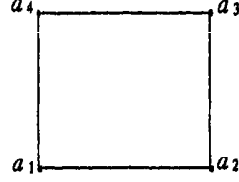


Fig. 1.1.5

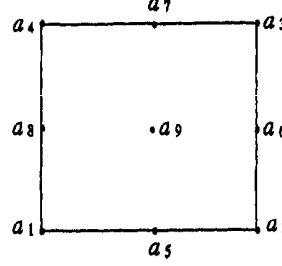


Fig. 1.1.6

where

$$\mu_1 = (1 - \xi)(1 - \eta), \mu_2 = \xi(1 - \eta), \mu_3 = \xi\eta, \mu_4 = (1 - \xi)\eta.$$

The corresponding finite element space $V_h \subset C(\bar{\Omega}) \cap H^1(\Omega)$.

Example 6 Lagrange bi-quadratic element (Fig. 1.1.6).

$K = \text{rectangle } a_1a_2a_3a_4$, midpoints a_5, a_6, a_7, a_8 , barycenter a_9 ;

$P_K = Q_2(K)$, $\dim P_K = 9$;

$\Sigma_K = \{p(a_i), 1 \leq i \leq 9\}$;

$$\forall p \in P_K, \quad p = \sum_{i=1}^9 p(a_i)\mu_i,$$

where

$$\mu_1 = (2\xi - 1)(\xi - 1)(2\eta - 1)(\eta - 1),$$

$$\mu_2 = \xi(2\xi - 1)(2\eta - 1)(\eta - 1),$$

$$\mu_3 = \xi(2\xi - 1)\eta(2\eta - 1),$$

$$\mu_4 = (2\xi - 1)(\xi - 1)\eta(2\eta - 1),$$

$$\mu_5 = 4\xi(1 - \xi)(2\eta - 1)(\eta - 1),$$

$$\mu_6 = 4\xi(2\xi - 1)\eta(1 - \eta),$$

$$\mu_7 = 4\xi(1 - \xi)\eta(2\eta - 1),$$

$$\begin{aligned}\mu_8 &= 4(2\xi - 1)(\xi - 1)\eta(1 - \eta), \\ \mu_9 &= 16\xi(1 - \xi)\eta(1 - \eta).\end{aligned}$$

The corresponding finite element space $V_h \subset C(\overline{\Omega}) \cap H^1(\Omega)$.

Example 7 Hermite bi-cubic element (Bogner-Fox-Schmidt rectangular element).

$$K = \text{rectangle } a_1 a_2 a_3 a_4;$$

$$P_K = \mathcal{Q}_3(K), \dim P_K = 16;$$

$$\Sigma_K = \left\{ p(a_i), \frac{\partial p(a_i)}{\partial x}, \frac{\partial p(a_i)}{\partial y}, \frac{\partial^2 p(a_i)}{\partial x \partial y}, 1 \leq i \leq 4 \right\}.$$

The corresponding finite element space $V_h \subset C^1(\overline{\Omega}) \cap H^2(\Omega)$.

Example 8 Hermite incomplete cubic element (Adini rectangular element).

$$K = \text{rectangle } a_1 a_2 a_3 a_4;$$

$$\begin{aligned}P_K &= \{p = p_3(x, y) + \alpha x^3 y + \beta x y^3 : \\ &\quad p_3(x, y) \in \mathcal{P}_3(K), \alpha, \beta \in \mathbb{R}\}, \dim P_K = 12;\end{aligned}$$

$$\Sigma_K = \left\{ p(a_i), \frac{\partial p(a_i)}{\partial x}, \frac{\partial p(a_i)}{\partial y}, 1 \leq i \leq 4 \right\}.$$

The corresponding finite element space $V_h \subset C(\overline{\Omega}) \cap H^1(\Omega)$.

In the above examples, the set of freedoms Σ_K is composed of the following "interpolation functionals":

$$l_i^{(\alpha)} : p \rightarrow D^\alpha p(a_i),$$

where a_i 's are nodes of the finite element, $\alpha = (\alpha_1, \alpha_2)$, and the orders of the derivatives $|\alpha| = 0, 1, 2$. Other kinds of freedoms can also be considered. For more examples of finite elements see, e.g., [A-2] and [B-17].

1.1.5 Interpolation error estimates in Sobolev spaces

The main results of the interpolation theory in Sobolev spaces are given in this subsection.

Definition 1.1.8 For a given finite element $\{K, P_K, \Sigma_K\}$, we call $\Pi_K v$ a P_K -interpolation of $v \in C^s(K)$ (where s is the highest order of the partial derivatives in Σ_K), if

$$\Pi_K v \in P_K,$$

$$l(\Pi_K v) = l(v), \forall l \in \Sigma_K.$$

$\Pi_K : C^s(K) \rightarrow P_K$ is then called a P_K -interpolation operator. Let V_h be the finite element space related to the grid $T_h = \{K\}$. We define $\Pi_h v$, the V_h -interpolation of $v \in C^s(\bar{\Omega})$, by

$$\Pi_h v \in V_h,$$

$$\Pi_h v|_K = \Pi_K v, \forall K \in T_h.$$

$\Pi_h : C^s(\bar{\Omega}) \rightarrow V_h$ is referred to as a V_h -interpolation operator.

Now our main task is to provide the error estimates, in the norm $\|\cdot\|_{m,q,K}$, of $v \in W^{k+1,p}(K)$ and its P_K -interpolation $\Pi_K v$, under the imbedding conditions $W^{k+1,p}(K) \rightarrow C^s(K)$ and $W^{k+1,p}(K) \rightarrow W^{m,q}(K)$. First we show a relationship between the norm and semi-norm of quotient spaces.

Theorem 1.1.10 Take a quotient norm

$$\|\dot{v}\|_{k+1,p,\Omega} \equiv \inf_{p \in \mathcal{P}_k} \|v + p\|_{k+1,p,\Omega}, \dot{v} \in W^{k+1,p}(\Omega)/\mathcal{P}_k,$$

in the quotient space $W^{k+1,p}(\Omega)/\mathcal{P}_k$, where v is any element in the equivalent class $\dot{v}$. Then there exists a constant $C = C(\Omega)$ such that

$$\|\dot{v}\|_{k+1,p,\Omega} \leq C|\dot{v}|_{k+1,p,\Omega}, \forall \dot{v} \in W^{k+1,p}(\Omega)/\mathcal{P}_k.$$

An easy consequence of Theorem 1.1.10 is the following abstract estimation for linear operators invariant with polynomials of degree at most k .

Theorem 1.1.11 Let Ω be a bounded open set with a Lipschitz continuous boundary. If

$$(i) \quad W^{k+1,p}(\Omega) \rightarrow W^{m,q}(\Omega),$$

(ii) Π is a bounded linear operator from $W^{k+1,p}(\Omega)$ to $W^{m,q}(\Omega)$, and is invariant on $\mathcal{P}_k$:

$$\Pi p = p, \quad \forall p \in \mathcal{P}_k,$$

then there exists a constant $C = C(\Omega)$ such that

$$|u - \Pi v|_{m,q,\Omega} \leq C \|I - \Pi\| \cdot |v|_{k+1,p,\Omega}, \quad \forall v \in W^{k+1,p}(\Omega).$$

The constants C and $\|I - \Pi\|$ in the above estimate depend on the region Ω and the operator Π respectively. To obtain an error estimate for a family of finite elements, we need to relate these finite elements with a special finite element through an affine mapping.

Definition 1.1.9 Two finite elements $\{\hat{K}, \hat{P}, \hat{\Sigma}\}$ and $\{K, P, \Sigma\}$ are said to be (affine) equivalent, if there exists an invertible affine mapping $F : \hat{x} \in \hat{K} \rightarrow x = F(\hat{x}) \in K$ satisfying

$$\begin{aligned} K &= F(\hat{K}), \\ P &= \{p = \hat{p} \circ F^{-1}, \hat{p} \in \hat{P}\}, \\ \Sigma &= \{l_i : p \rightarrow \hat{l}_i(p), \forall p = \hat{p} \circ F^{-1}, \hat{l}_i \in \hat{\Sigma}\}. \end{aligned}$$

A family of finite elements is referred to as an affine family, if all the finite elements in the family are (affine) equivalent to a certain finite element, called the reference element of the family. An affine family is said to be regular if there exists a constant σ such that for all K

$$h_K/\rho_K \leq \sigma, \text{ and } h_K \rightarrow 0,$$

where $h_K = \text{diam}(K)$, $\rho_K = \sup\{\text{diam}(S) : \text{the ball } S \subset K\}$.

The next result reveals a relationship between the Sobolev seminorms of a function before and after an affine mapping.

Theorem 1.1.12 Let $\hat{\Omega}$ and Ω be two affine equivalent open sets in R^n , i.e., there is an invertible affine mapping

$$F : \hat{x} \in \hat{\Omega} \rightarrow F(\hat{x}) = B\hat{x} + b \in \Omega,$$

such that

$$\Omega = F(\hat{\Omega}),$$

where B is an $n \times n$ nonsingular matrix and b an n -dimensional vector. Then there exists a constant $C = C(m, n, p)$ independent of Ω and F such that for any $v \in W^{m,p}(\Omega)$

$$|\hat{v}|_{m,p,\hat{\Omega}} \leq C \|B\|^m |\det B|^{-1/p} |v|_{m,p,\Omega},$$

where $\hat{v}(\hat{x}) = v(F(\hat{x}))$. Conversely, for any $\hat{v} \in W^{m,p}(\hat{\Omega})$

$$\|v\|_{m,p,\Omega} \leq C \|B^{-1}\|^m |\det B|^{1/p} |\hat{v}|_{m,p,\hat{\Omega}},$$

where $v(x) = \hat{v}(F^{-1}(x))$. Besides,

$$\|B\| \leq h_{\Omega}/\rho_{\hat{\Omega}}, \quad \|B^{-1}\| \leq h_{\hat{\Omega}}/\rho_{\Omega},$$

$$|\det B| = \text{meas}(\Omega)/\text{meas}(\hat{\Omega}),$$

where $h_{\Omega} = \text{diam}\Omega$, $\rho_{\Omega} = \sup\{\text{diam}S : \text{the ball } S \subset \Omega\}$.

Transfer, by virtue of Theorem 1.1.12, the estimate for the interpolation error $v - \Pi_K v$ of the finite element $\{K, P, \Sigma\}$ to the reference element $\{\hat{K}, \hat{P}, \hat{\Sigma}\}$, and apply the abstract estimate given in Theorem 1.1.11 to the reference element, then we have the following important result.

Theorem 1.1.13 *For a given regular family of finite elements, suppose the reference element $\{\hat{K}, \hat{P}, \hat{\Sigma}\}$ satisfies*

$$W^{k+1,p}(\hat{K}) \rightarrow C^s(\hat{K}),$$

$$W^{k+1,p}(\hat{K}) \rightarrow W^{m,q}(\hat{K}),$$

$$\mathcal{P}_k \subset \hat{P} \subset W^{m,q}(\hat{K}),$$

where s is the highest order of the derivatives in $\hat{\Sigma}$; m, k are non-negative integers; and $1 \leq p, q \leq \infty$. Then there exists a constant C independent of K such that for every finite element K in the family and any function $v \in W^{k+1,p}(K)$

$$|v - \Pi_K v|_{m,p,K} \leq C (h_K^n)^{\frac{1}{q} - \frac{1}{p}} h_K^{k+1-m} |v|_{k+1,p,K}.$$

In particular, if $p = q = 2$ then

$$|v - \Pi_K v|_{m,K} \leq C h_K^{k+1-m} |v|_{k+1,K}.$$

Finally we present an inverse property of the finite element. To this end we need further assumptions for the grid.

Definition 1.1.10 *A family of grids $\{T_h\}$ is said to be quasi-uniform if there are constants σ and γ such that*

$$h_K/\rho_K \leq \sigma, \quad h/h_K \leq \gamma, \quad \forall K \in T_h, \quad h > 0.$$

Theorem 1.1.14 (Inverse property) *Let a family of finite elements with quasi-uniform grids be given. Assume that V_h is a finite element space related to a grid T_h , that l, m are nonnegative integers, that $1 \leq r, q \leq \infty$, and that*

$$l \leq m, \quad \hat{P} \subset W^{l,r}(\hat{K}) \cap W^{m,q}(\hat{K}).$$

Then there exists a constant $C = C(\sigma, \gamma, l, m, r, q)$ such that for all $v_h \in V_h$

$$\left(\sum_{K \in T_h} |v_h|_{m,q,K}^q \right)^{1/q} \leq \frac{C}{(h^n)^{\max\{0, \frac{1}{r} - \frac{1}{q}\}} h^{m-l}} \left(\sum_{K \in T_h} |v_h|_{l,r,K}^r \right)^{1/r}.$$

Here we make a convention for $q = \infty$ that

$$\left(\sum_{K \in T_h} |v_h|_{m,q,K}^q \right)^{1/q} = \max_{K \in T_h} |v_h|_{m,\infty,K}.$$

In particular, when $r = q = 2$ we have

$$\left(\sum_{K \in T_h} |v_h|_{m,K}^2 \right)^{1/2} \leq Ch^{l-m} \left(\sum_{K \in T_h} |v_h|_{l,K}^2 \right)^{1/2}.$$

1.2 Variational Problems and Their Approximations

1.2.1 Abstract variational form

Let H be a real Hilbert space, equipped with an inner product $(\cdot, \cdot)$ and the corresponding norm $\|\cdot\|$. V is a subspace of H satisfying

$\bar{V} = H$ according to the norm $\|\cdot\|$. V becomes a Hilbert space with respect to an inner product $[\cdot, \cdot]$ and its related norm $|\cdot|$. The imbedding of V in H is continuous, that is, there exists a constant $\gamma > 0$ such that

$$\|v\| \leq \gamma|v|, \quad \forall v \in V.$$

Suppose A is a linear operator from a dense linear subspace $D(A)$ of V to the dual space V' of V , satisfying $\overline{D(A)} = V$ (in the norm $|\cdot|$). For $f \in V'$ consider an operator equation

$$Au = f. \quad (1.2.1)$$

In many cases, equation (1.2.1) does not necessarily have a solution in $D(A)$. Thus we need to extend the operator A in some sense and then to discuss the generalized solution of the problem. To this end, we construct a bilinear form:

$$a(u, v) \equiv \langle Au, v \rangle = Au(v), \quad \forall u, v \in D(A)$$

(where the notation $\langle \cdot, \cdot \rangle$ denotes the dual pair of $V' \times V$), and make the following basic hypothesis:

(H) $a(u, v)$ is a bounded bilinear form on $D(A) \times D(A)$, i.e., there exists a constant M such that

$$|a(u, v)| \leq M|u||v|, \quad \forall u, v \in D(A).$$

Now we consider the problem in the space V . For any $u \in D(A)$, $\langle Au, v \rangle$ is a bounded linear functional of v on V . By the Riesz representation theorem there is a unique element $Au \in V$ satisfying

$$\langle Au, v \rangle = [Au, v], \quad u \in D(A), \quad v \in V.$$

Similarly for each $f \in H$ we have a unique $Rf \in V$ such that

$$\langle f, v \rangle = [Rf, v], \quad v \in V.$$

Proposition 1.2.1 *Let (H) hold. Then $\mathcal{A} : D(A) \rightarrow V$ can be uniquely extended into a bounded linear operator $T : V \rightarrow V$, and*

the corresponding bilinear form can be (uniquely and continuously) extended onto $V \times V$:

$$a(u, v) = [Tu, v], \quad u, v \in V,$$

and we have

$$|a(u, v)| \leq M|u| |v|, \quad \forall u, v \in V.$$

Proof For any $u \in V$, if $u \in D(A)$, then set $Tu = Au$; If $u \notin D(A)$, set $Tu = \lim_{j \rightarrow \infty} Au_j$ (in V), where $\{u_j\} \subset D(A)$, $\lim_{j \rightarrow \infty} u_j = u$ (in V).

Such a sequence $\{u_j\}$ does exist since $\overline{D(A)} = V$ (according to $|\cdot|$), and $\{Au_j\}$ converges in V by condition (H). The limit Tu is obviously independent of the choice of $\{u_j\}$. It is easy to check that the bilinear form remains to be bounded after the extension, and hence T is a bounded linear operator:

$$|Tu| = \sup_{v \in V, |v| \leq 1} |[Tu, v]| \leq M|u|, \quad \forall u \in V.$$

Finally, this continuous extension is apparently unique. This completes the proof. $\square$

Now in place of (1.2.1), we consider the operator equation

$$Tu = Rf,$$

or equivalently

$$[Tu, v] = [Rf, v], \quad \forall v \in V,$$

that is,

$$a(u, v) = \langle f, v \rangle, \quad \forall v \in V. \quad (1.2.2)$$

Definition 1.2.1 The solution $u \in V$ of the equation (1.2.2) is called the Galerkin generalized (or weak) solution of the original equation (1.2.1), and the solution of (1.2.1) in $D(A)$ the classical solution.

The following conclusion is obvious.

Theorem 1.2.1 If $u \in D(A)$ is a classical solution of (1.2.1), then it is also a Galerkin generalized solution. Conversely, if u is a Galerkin generalized solution of (1.2.1) and $u \in D(A)$, then it is also a classical solution.

The above result is called a variational principle in Galerkin form, and equation (1.2.2) a variational problem in Galerkin form with respect to (1.2.1). In mechanics, v stands for the virtual displacement, $a(u, v) - \langle f, v \rangle$ the virtual work, and (1.2.2) the virtual work equation. So we also call Theorem 1.2.1 a virtual work principle.

Assume that $a(\cdot, \cdot)$ is symmetric. Let us introduce a quadratic functional

$$J(v) = \frac{1}{2}a(v, v) - \langle f, v \rangle, \quad \forall v \in V,$$

and consider the following functional minimization problem: Find $u \in V$ such that

$$J(u) = \inf_{v \in V} J(v). \quad (1.2.3)$$

Many practical problems (e.g. the elastic problems) can be deduced into this form.

Definition 1.2.2 *The solution $u \in V$ of the problem (1.2.3) is referred to as a Riesz generalized (or weak) solution of the original equation (1.2.1).*

Theorem 1.2.2 *Assume that $a(\cdot, \cdot)$ is symmetric and positive definite:*

$$a(u, v) = a(v, u), \quad \forall u, v \in V, \quad (1.2.4)$$

$$a(v, v) \geq \alpha|v|^2, \quad \forall v \in V \quad (\alpha \text{ a positive constant}). \quad (1.2.5)$$

Then the problems (1.2.3) and (1.2.2) are equivalent. So under the above assumptions, if $u \in D(A)$ is a classical solution of equation (1.2.1), then it is also a Riesz generalized solution; and conversely, if u is a Riesz generalized solution of (1.2.1) and $u \in D(A)$, then it is a classical solution.

Remark It is the condition (1.2.5) that is called the V -elliptic condition, also often referred to as a coercive, or positive definite condition.

Proof We consider for $u, v \in V$

$$\begin{aligned}\phi(t) &\equiv J(u + tv) \\ &= \frac{1}{2}a(u + tv, u + tv) - \langle f, u + tv \rangle \\ &= J(u) + t[a(u, v) - \langle f, v \rangle] + \frac{t^2}{2}a(v, v).\end{aligned}$$

The symmetry condition (1.2.4) is used in the last equality. It follows from the positive definite condition (1.2.5) that u is the minimum function of the functional $J(u)$ if and only if $\phi'(0) = 0$ i.e.

$$a(u, v) = \langle f, v \rangle, \quad \forall v \in V.$$

The second half of the theorem results from Theorem 1.2.1. This completes the proof. $\square$

Theorem 1.2.2 is called a Riesz variational principle, and problem (1.2.3) a Riesz variational problem corresponding to (1.2.1). In mechanics, the quadratic functional $J(u)$ represents the energy of the system. The above conclusion illustrates that in all the possible displacements satisfying the given boundary constraints, the displacement that makes the balance minimizes the total potential energy. Therefore, Theorem 1.2.2 is also called the minimum potential energy principle.

The virtual work principle is more general and has wider applications than the potential energy principle. It applies not only to symmetric and positive definite problems (corresponding to conservative field equations) but also to asymmetric and nonpositive definite problems (nonconservative field equations).

1.2.2 Green's formulas and variational problems

For differential equations, the continuous extension of the operators and the bilinear forms as well as the deduction of the variational forms mentioned in the above subsection are realized by integrations in parts or by the use of Green's formulas.

Let Ω be a bounded open region in R^N with a Lipschitz continuous boundary $\Gamma = \partial\Omega$, and $n = (n_1, \dots, n_N)$ the unit outer normal vector.

Then the following Green's formula holds:

$$-\int_{\Omega} \frac{\partial u}{\partial x_i} v dx = \int_{\Omega} u \frac{\partial v}{\partial x_i} dx - \int_{\Gamma} u v n_i ds, \quad \forall u, v \in H^1(\Omega). \quad (1.2.6)$$

Replacing u by $\frac{\partial u}{\partial x_i}$ and summing for i lead to the first Green's formula:

$$-\int_{\Omega} \Delta u v dx = \int_{\Omega} \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx - \int_{\Gamma} \frac{\partial u}{\partial n} v ds, \\ \forall u \in H^2(\Omega), v \in H^1(\Omega). \quad (1.2.7)$$

Exchange the positions of u and v to get another equality, and subtract the above equality from it, then we have the second Green's formula:

$$\int_{\Omega} (v \Delta u - u \Delta v) dx = \int_{\Gamma} \left(\frac{\partial u}{\partial n} v - \frac{\partial v}{\partial n} u \right) ds, \quad \forall u, v \in H^2(\Omega). \quad (1.2.8)$$

If we replace u above by Δu , then we obtain another Green's formula:

$$\int_{\Omega} v \Delta^2 u dx = \int_{\Omega} \Delta u \Delta v dx + \int_{\Gamma} \left(\frac{\partial \Delta u}{\partial n} v - \Delta u \frac{\partial v}{\partial n} \right) ds, \\ \forall u \in H^4(\Omega), v \in H^2(\Omega). \quad (1.2.9)$$

It is easy to show for $N = 2$ that

$$\int_{\Omega} \left(2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v}{\partial x_1^2} \right) dx_1 dx_2 \\ = \int_{\Gamma} \left(-\frac{\partial^2 u}{\partial \tau^2} \frac{\partial v}{\partial n} + \frac{\partial^2 u}{\partial \tau \partial n} \frac{\partial v}{\partial \tau} \right) ds, \quad \forall u \in H^3(\Omega), v \in H^2(\Omega). \quad (1.2.10)$$

Here $\tau = (\tau_1, \tau_2)$ is the unit tangent vector along Γ , $\frac{\partial}{\partial \tau}$ the derivative along the tangent direction, and

$$\frac{\partial^2 u}{\partial \tau^2} = D^2 u \cdot (\tau, \tau) = \sum_{i,j=1}^2 \tau_i \tau_j \frac{\partial^2 u}{\partial x_i \partial x_j}, \\ \frac{\partial^2 u}{\partial \tau \partial n} = D^2 u \cdot (\tau, n) = \sum_{i,j=1}^2 \tau_i n_j \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

As an example of variational problems, let us consider the mixed boundary value problem of the Poisson equation:

$$\begin{cases} -\Delta u = f(x, y), & (x, y) \in \Omega, \end{cases} \quad (1.2.11a)$$

$$\begin{cases} u|_{\Gamma_1} = 0, \end{cases} \quad (1.2.11b)$$

$$\begin{cases} \left(\frac{\partial u}{\partial n} + \alpha u \right) \Big|_{\Gamma_2} = 0, \end{cases} \quad (1.2.11c)$$

where Ω is a bounded plane region. Its boundary Γ is a piecewise smooth simple closed curve, divided into two disjoint parts Γ_1 and Γ_2 . $\alpha > 0$ and $f \in L^2(\Omega)$.

Set

$$V = H_E^1(\Omega) = \{v : v \in H^1(\Omega), v|_{\Gamma_1} = 0\},$$

$$a(u, v) = \iint_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx dy + \int_{\Gamma_2} \alpha uv ds,$$

$$\langle f, v \rangle = \iint_{\Omega} f v dx dy.$$

Multiply (1.2.11a) by $v \in V$, integrate it on Ω , and employ Green's formula (1.2.7) and the boundary conditions (1.2.11b,c) to obtain

$$\iint_{\Omega} -v \Delta u dx dy = \iint_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx dy + \int_{\Gamma_2} \alpha uv ds.$$

Now the variational problem corresponding to problem (1.2.11) becomes: Find $u \in V$ such that

$$a(u, v) = \langle f, v \rangle, \quad \forall v \in V. \quad (1.2.12)$$

Here $a(u, v)$ is a continuous extension of $(-\Delta u, v)$ thanks to Green's formula. This kind of continuous extension on $V \times V$ is unique and hence (1.2.12) is identical with the variational form mentioned in the above subsection.

Note that $|\cdot|_1$ is an equivalent norm of H_E^1 , and that $a(\cdot, \cdot)$ is symmetric and positive definite:

$$a(u, v) = a(v, u), \quad \forall u, v \in V,$$

$$a(v, v) \geq |v|_1^2, \quad \forall v \in V.$$

As an example of high order equations, we consider the first boundary value problem of the biharmonic equation:

$$\begin{cases} \Delta^2 u = \frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = f, & (x, y) \in \Omega, & (1.2.13a) \\ u|_{\Gamma} = 0, & & (1.2.13b) \\ \frac{\partial u}{\partial n}|_{\Gamma} = 0, & & (1.2.13c) \end{cases}$$

where $f \in L^2(\Omega)$.

It follows from Green's formula (1.2.9) that

$$\iint_{\Omega} v \Delta^2 u dx dy = \iint_{\Omega} \Delta u \Delta v dx dy, \quad v \in H_0^2(\Omega).$$

Write

$$\begin{aligned} V &= H_0^2(\Omega), \\ a(u, v) &= \iint_{\Omega} \Delta u \Delta v dx dy, \\ \langle f, v \rangle &= \iint_{\Omega} f v dx dy. \end{aligned}$$

Then the variational problem related to (1.2.13) becomes: Find $u \in V$ such that

$$a(u, v) = \langle f, v \rangle, \quad \forall v \in V. \quad (1.2.14)$$

These kind of problems arise particularly in fluid mechanics.

Obviously $a(\cdot, \cdot)$ is a symmetric bilinear form. As for its positive definiteness, we first note

$$a(v, v) = |\Delta v|_{0,\Omega}^2 = \iint_{\Omega} \left[\left(\frac{\partial^2 v}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 v}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 v}{\partial y^2} \right)^2 \right] dx dy.$$

Apply twice Green's formula (1.2.6), then we have

$$\begin{aligned} \iint_{\Omega} \left(\frac{\partial^2 v}{\partial x \partial y} \right)^2 dx dy &= - \iint_{\Omega} \frac{\partial v}{\partial x} \frac{\partial^3 v}{\partial x \partial y^2} dx dy \\ &= \iint_{\Omega} \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 v}{\partial y^2} dx dy, \quad \forall v \in C_0^\infty(\Omega). \end{aligned}$$

The density of $C_0^\infty(\Omega)$ in $H_0^2(\Omega)$ implies that

$$\iint_{\Omega} \left(\frac{\partial^2 v}{\partial x \partial y} \right)^2 dx dy = \iint_{\Omega} \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 v}{\partial y^2} dx dy, \quad \forall v \in H_0^2(\Omega).$$

So we have

$$a(v, v) = |\Delta v|_{0,\Omega}^2 = |v|_{2,\Omega}^2, \quad \forall v \in H_0^2(\Omega).$$

Now the positive definiteness of $a(\cdot, \cdot)$ follows since $|\cdot|_{2,\Omega}$ is an equivalent norm on $H_0^2(\Omega)$.

By Green's formula (1.2.10) and boundary conditions (1.2.13b,c), the above bilinear form can be rewritten as

$$\begin{aligned} a(u, v) &= \iint_{\Omega} \left[\Delta u \Delta v + (1-\sigma) \left(2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial x^2} \right) \right] dx dy \\ &= \iint_{\Omega} \left[\sigma \Delta u \Delta v + (1-\sigma) \left(2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial y^2} \right) \right] dx dy, \end{aligned}$$

where σ is the Poisson ratio satisfying $0 < \sigma < \frac{1}{2}$. This corresponds to the variational form for the clamped plate problem and it also applies to the plate bending with other boundary constraints. The bilinear form is again symmetric and positive definite:

$$a(v, v) = \sigma |\Delta v|_{0,\Omega}^2 + (1-\sigma) |v|_{2,\Omega}^2.$$

1.2.3 Well-posedness of variational problems

The following important and widely used result concerns the well-posedness of variational problems, namely the solution's existence, uniqueness and continuous dependence on the right-hand side term.

Theorem 1.2.3 (Lax-Milgram) *Let V be a real Hilbert space and $a(\cdot, \cdot)$ a bilinear form defined on $V \times V$, satisfying the following conditions:*

(i) *Boundedness: There exists a constant $M > 0$ such that*

$$|a(u, v)| \leq M |u| |v|, \quad \forall u, v \in V.$$

(ii) *Positive definiteness: There exists a constant $\alpha > 0$ such that*

$$|a(v, v)| \geq \alpha |v|^2, \quad \forall v \in V.$$

Then, the variational problem (1.2.2) has a unique solution $u \in V$ for any given $f \in V'$ and

$$|u| \leq \frac{1}{\alpha} \|f\|_{V'}.$$

The next theorem is a generalization of Theorem 1.2.3.

Theorem 1.2.4 (Babuska) *Let U and V be two real Hilbert spaces, supplied with inner products $[\cdot, \cdot]_U$, $[\cdot, \cdot]_V$ and norms $|\cdot|_U$, $|\cdot|_V$ respectively. $a(\cdot, \cdot)$ is a bilinear form on $U \times V$, satisfying the following conditions:*

(i) *Boundedness: There exists a constant $M > 0$ such that*

$$|a(u, v)| \leq M |u|_U |v|_V, \quad \forall u \in U, v \in V. \quad (1.2.15)$$

(ii) *Weak positive definiteness: There exists a constant $\alpha > 0$ such that*

$$\inf_{\substack{u \in U \\ |u|_U=1}} \sup_{\substack{v \in V \\ |v|_V=1}} |a(u, v)| \geq \alpha, \quad (1.2.16)$$

$$\sup_{u \in U} |a(u, v)| > 0, \quad \forall v \in V, v \neq 0. \quad (1.2.17)$$

Then, for any given $f \in V'$ there exists a unique $u \in U$ such that

$$a(u, v) = \langle f, v \rangle, \quad \forall v \in V, \quad (1.2.18)$$

and

$$|u|_U \leq \frac{1}{\alpha} \|f\|_{V'}. \quad (1.2.19)$$

Proof For any $u \in U$, the boundedness guarantees that $a(u, \cdot)$ is a bounded linear functional on V . So by the Riesz representation theorem there is a $Tu \in V$ such that

$$a(u, v) = [Tu, v]_V, \quad \forall v \in V.$$

Similarly for $f \in V'$ there exists an $Rf \in V$ such that

$$\langle f, v \rangle = [Rf, v]_V, \quad \forall v \in V.$$

Therefore, (1.2.18) is equivalent to a bounded linear operator equation:

$$Tu = Rf. \quad (1.2.20)$$

The weak positive definiteness implies

$$\inf_{\substack{u \in U \\ \|u\|_U=1}} |Tu|_V \geq \alpha > 0,$$

which indicates that the operator T has a bounded inverse operator T^{-1} satisfying

$$\|T^{-1}\| \leq \alpha^{-1}.$$

In particular, the range of T is a closed linear subspace of V .

Now we claim that the range of T is indeed the whole space V . If this is not true, then by virtue of the projection theorem there must be a $v_0 \in V, v_0 \neq 0$ satisfying

$$[Tu, v_0] = a(u, v_0) = 0, \quad \forall u \in U.$$

But this contradicts (1.2.17). Therefore, (1.2.18) or (1.2.20) must have a unique solution $u = T^{-1}Rf$. Finally (1.2.19) holds since

$$\begin{aligned} \|u\|_U &\leq \alpha^{-1} \|Rf\|_V = \alpha^{-1} \sup_{\substack{v \in V \\ \|v\|_V=1}} |[Rf, v]_V| \\ &= \alpha^{-1} \sup_{\substack{v \in V \\ \|v\|_V=1}} |\langle f, v \rangle| = \alpha^{-1} \|f\|_{V'}. \end{aligned}$$

This completes the proof. $\square$

1.2.4 Approximation methods. A necessary and sufficient condition for approximate-solvability

Next we discuss the approximation methods. Notice that the abstract variational problem is equivalent to a bounded linear operator

equation. So in this subsection, we discuss in a more general setting the approximation methods for bounded linear operator equations and present a necessary and sufficient condition for the unique approximate-solvability.

Suppose U and V are two reflexive Banach spaces. We denote both the norms by $\|\cdot\|$. Let T be a bounded linear operator from U to V . We consider the approximation methods for the operator equation

$$Tu = g, \quad (1.2.21)$$

where $g \in V$ is given.

The essence of the finite dimensional approximation is to discretize the problem and replace the original equation by an approximate equation in a finite dimensional space. First we need to discretize the solution space and the range space, and accordingly to construct an approximation of the original equation. Different strategies of discretization and construction lead to different numerical methods. Generally, we will choose in a certain way finite dimensional subspaces U_n and V_n of U and V , and the mappings $P_n : U \rightarrow U_n$ and $Q_n : V \rightarrow V_n$, respectively. Then set $T_n = Q_n T|_{U_n}$ and take

$$T_n u_n = Q_n g \quad (u_n \in U_n) \quad (1.2.22)$$

as the approximation equation of (1.2.21).

P_n and Q_n are linear operators for most of the approximation methods. If we choose P_n and Q_n as linear projection operators, then we end up with the so-called projection methods. They become the usual Galerkin methods if $U = V$, $U_n = V_n$. If T is a differential or integral operator, and U_n is a spline function space (or a piecewise polynomial space), then we have the finite element method. If U and V are function spaces on Ω , and $Q_n : V \rightarrow V_n$ is an interpolation operator:

$$Q_n v(x) = \sum_{i=1}^N v(x_i) \psi_i(x), \quad v \in V,$$

where $x_1, \dots, x_N \in \Omega$, and $\{\psi_1, \dots, \psi_N\}$ is a basis of V_n , then the

approximation equation (1.2.22) reads

$$\sum_{i=1}^N (Tu_n)_{x=x_i} \psi_i(x) = \sum_{i=1}^N (g)_{x=x_i} \psi_i(x),$$

which is equivalent to

$$(Tu_n)_{x=x_i} = (g)_{x=x_i}, \quad i = 1, 2, \dots, N. \quad (1.2.23)$$

This is the so-called collocation method. If U and V are Hilbert spaces; $(\cdot, \cdot)$ is the inner product of V ; $U_n = \text{span}\{\phi_1, \dots, \phi_N\}$ and $V_n = \text{span}\{\psi_1, \dots, \psi_N\}$; P_n and Q_n are orthogonal projection operators; then the approximation equation (1.2.22) is equivalent to

$$(Tu_n - g, \psi_i) = 0, \quad i = 1, 2, \dots, N, \quad (1.2.24)$$

which is the Petrov-Galerkin method, also called the generalized Galerkin method. There are many different choices for $\{\phi_i\}$ and $\{\psi_i\}$. Choosing $\psi_i = M\phi_i$ for some suitably chosen linear operator M leads to a moment method. If in particular $\psi_i = T\phi_i$, then it becomes the least square method, since in this case (1.2.24) is equivalent to the problem of finding $u \in U_n$ to minimize $\|Tu_n - g\|$. The case of $U = V$, $\psi_i = \phi_i$ corresponds to the Galerkin method.

Now we turn to discuss the approximate-solvability. Let $\{U_n\}$ and $\{V_n\}$ be sequences of finite dimensional subspaces of U and V , respectively. $P_n : U \rightarrow U_n$ and $Q_n : V \rightarrow V_n$ are respectively the linear projection operators, satisfying $P_n P_m = P_n$ and $Q_n Q_m = Q_n$ for $n \leq m$ and the following properties:

- (1) $U_n \subset U_{n+1}$, $V_n \subset V_{n+1}$, $n = 1, 2, \dots$;
- (2) $\bigcup_{n=1}^{\infty} U_n = U$, $\bigcup_{n=1}^{\infty} V_n = V$;
- (3) $\|P_n\| \leq C$, $\|Q_n\| \leq C$, $n = 1, 2, \dots$ (C is a constant.)

Proposition 1.2.2 *Under the above assumptions we have*

- (i) $\forall u \in U$, $P_n u \rightarrow u$, as $n \rightarrow \infty$;
- (ii) $\forall v \in V$, $Q_n v \rightarrow v$, as $n \rightarrow \infty$;
- (iii) $\forall l \in U'$, $P'_n l \rightarrow l$, as $n \rightarrow \infty$;

(iv) $\forall l \in V', Q'_n l \rightarrow l$, as $n \rightarrow \infty$;
 where U' and V' are the dual spaces of U and V , and P'_n and Q'_n are the dual operators of P_n and Q_n , respectively.

Proof To show (i), we note from conditions (1) and (2) that for any $u \in U$ there exists a $u_n \in U_n$ ($n = 1, 2, \dots$) satisfying

$$\|u - u_n\| \rightarrow 0, n \rightarrow \infty.$$

Then it follows from the triangular inequality and condition (3) that

$$\begin{aligned} \|P_n u - u\| &\leq \|P_n(u - u_n)\| + \|u_n - u\| \\ &\leq (C + 1)\|u_n - u\| \rightarrow 0, n \rightarrow \infty. \end{aligned}$$

Now we deal with (iii). Write $W_n = P'_n U'$. It results from $P_n P_m = P_n$ when $n \leq m$ that $P'_m P'_n = P'_n$. Hence $W_n \subset W_m$ ($n \leq m$), and in particular $W_n \subset W_{n+1}$.

By the reflexivity of the space U , if $\overline{\bigcup_{n=1}^{\infty} W_n} = U'$ does not hold, then there exists a $u_0 \in U$, $u_0 \neq 0$ such that

$$\langle l, u_0 \rangle = 0, \forall l \in \bigcup_{n=1}^{\infty} W_n.$$

So for any $l \in U'$ and $n \geq 1$ we have

$$\langle l, P_n u_0 \rangle = \langle P'_n l, u_0 \rangle = 0,$$

which implies $P_n u_0 = 0$. But $P_n u_0 \rightarrow u_0$ as $n \rightarrow \infty$. So $u_0 = 0$, yielding a contradiction. Therefore we must have $\overline{\bigcup_{n=1}^{\infty} W_n} = U'$.

It follows from condition (3) that $\|P'_n\| \leq C$. Thus, we can show (iii) as in the proof to (i).

(ii) and (iv) can be similarly proved. This completes the proof. $\square$

Proposition 1.2.3 Under the above conditions we have ($\rightharpoonup$ stands for the weak convergence):

- (i) If $\{u_j\} \subset U$ and $u_j \rightharpoonup u \in U$ ($j \rightarrow \infty$), then $Tu_j \rightharpoonup Tu \in V$.
- (ii) If $u \in U$, then $T_n P_n u \rightharpoonup Tu$ ($n \rightarrow \infty$).
- (iii) If $\{u_j\} \subset U$, $u_j \in U_{n_j}$ ($j = 1, 2, \dots$), $n_j \rightarrow \infty$ ($j \rightarrow \infty$); and $u_j \rightharpoonup u \in U$ ($j \rightarrow \infty$), then $T_{n_j} u_j \rightharpoonup Tu$ ($j \rightarrow \infty$).

Proof (i) For any $l \in V'$ and $u_j \rightarrow u$ ($j \rightarrow \infty$) we have

$$\langle l, Tu_j \rangle = \langle T'l, u_j \rangle \rightarrow \langle T'l, u \rangle = \langle l, Tu \rangle,$$

which means $Tu_j \rightarrow Tu$ ($j \rightarrow \infty$).

(ii) For $u \in U$ it follows from Proposition 1.2.2 that

$$\begin{aligned} & \|T_n P_n u - Tu\| \\ & \leq \|Q_n T(P_n u - u)\| + \|Q_n Tu - Tu\| \\ & \leq C\|T\|\|P_n u - u\| + \|Q_n Tu - Tu\| \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

(iii) We know by (i) that $Tu_j \rightarrow Tu$ ($j \rightarrow \infty$). By Proposition 1.2.2 we have for any $l \in V'$ that $Q'_{n_j} l \rightarrow l$ ($j \rightarrow \infty$). This gives

$$\langle l, T_{n_j} u_j \rangle = \langle Q'_{n_j} l, Tu_j \rangle \rightarrow \langle l, Tu \rangle \quad (j \rightarrow \infty).$$

This implies $T_{n_j} u_j \rightarrow Tu$ ($j \rightarrow \infty$) and completes the proof. $\square$

Definition 1.2.3 Equation (1.2.21) is said to be uniquely approximate-solvable, if there exists an integer $N > 0$ such that for $n \geq N$ equation (1.2.22) possesses a unique solution $u_n \in U_n$, and $\{u_n\}$ converges in U as $n \rightarrow \infty$, of which the limit $u \in U$ is the unique solution of (1.2.21).

Theorem 1.2.5 Let U and V be reflexive Banach spaces, $T : U \rightarrow V$ a bounded linear operator, and U_n, V_n, P_n, Q_n as above. Then, a sufficient and necessary condition for equation (1.2.21) to be uniquely approximate-solvable for any given $g \in V$ is that there exists an integer $N > 0$ and a constant $\alpha > 0$ such that

$$\|Q_n Tu\| \geq \alpha \|u\|, \quad \forall u \in U_n, \quad n \geq N, \quad (1.2.25)$$

or equivalently

$$\lim_{n \rightarrow \infty} \inf_{\substack{u \in U_n \\ u \neq 0}} \left\{ \frac{\|Q_n Tu\|}{\|u\|} \right\} > 0.$$

In this case, we have the following error estimate for u_n

$$\|u - u_n\| \leq \left(1 + \frac{C}{\alpha} \|T\|\right) \inf_{w \in U_n} \|u - w\|. \quad (1.2.26)$$

Proof Necessity. That for any $g \in V$ equation (1.2.22) always has a unique solution $u_n \in U_n$ means that there exists an inverse operator $T_n^{-1} : V_n \rightarrow U_n$ and the range of T_n is V_n . Moreover, T_n^{-1} is bounded by the inverse operator theorem. For the operator $T_n^{-1}Q_n : V \rightarrow U_n \subset U$, we note

$$\forall g \in V, T_n^{-1}Q_n g = u_n \rightarrow u \quad (n \rightarrow \infty),$$

where u_n and u are the solutions of (1.2.22) and (1.2.21) respectively. So it follows from the resonance theorem that the sequence of operators $\{T_n^{-1}Q_n\}$ is uniformly bounded, namely, there exists a constant $\beta > 0$ such that

$$\|T_n^{-1}Q_n\| \leq \beta \quad (n \leq N).$$

So (1.2.25) is valid:

$$\|T_n u\| \geq \frac{1}{\beta} \|u\|, \quad \forall u \in U_n, \quad n \geq N.$$

Sufficiency. Condition (1.2.25) implies that when $n \geq N$, for any $g \in V$ equation (1.2.22) has a unique solution $u_n \in U_n$ and

$$\|u_n\| \leq \frac{C}{\alpha} \|g\|.$$

The reflexivity of the space U gives the existence of a weakly convergent subsequence $\{u_{n_j}\}$ satisfying

$$u_{n_j} \in U_{n_j}, \quad u_{n_j} \rightharpoonup u \in U \quad (j \rightarrow \infty).$$

By (iii) of Proposition 1.2.3 we have

$$T_{n_j} u_{n_j} \rightarrow T u \quad (j \rightarrow \infty).$$

On the other hand

$$T_{n_j} u_{n_j} = Q_{n_j} g \rightarrow g \quad (j \rightarrow \infty).$$

Hence u is the solution to (1.2.21).

Let us show $u_n \rightarrow u$ ($j \rightarrow \infty$). In fact, by virtue of (1.2.25) for any $\tilde{u} \in U_n$

$$\begin{aligned} \|u_n - u\| &\leq \|u_n - \tilde{u}\| + \|\tilde{u} - u\| \\ &\leq \frac{1}{\alpha} \|Q_n T(u_n - \tilde{u})\| + \|\tilde{u} - u\| \\ &= \frac{1}{\alpha} \|Q_n T(u - \tilde{u})\| + \|u - \tilde{u}\| \\ &\leq \left(\frac{C}{\alpha} \|T\| + 1\right) \|u - \tilde{u}\|. \end{aligned}$$

This gives (1.2.26). Setting $\tilde{u} = P_n u$ yields $u_n \rightarrow u$ ($n \rightarrow \infty$).

Finally we claim that condition (1.2.25) guarantees the uniqueness of the solution to (1.2.21). In fact, if $w \in U$ satisfies $Tw = 0$, then it follows from (1.2.25) and (ii) of Proposition 1.2.3 that

$$\|P_n w\| \leq \frac{1}{\alpha} \|T_n P_n w\| \rightarrow 0 \quad (n \rightarrow \infty),$$

which gives $w = 0$. This completes the proof. $\square$

1.2.5 Galerkin methods

Let us recall the framework of the abstract variational problem in §1.2.1. Suppose H and V are the Hilbert spaces mentioned there, A is the linear operator from a dense subset $D(A)$ of V to V' , and $f \in V'$. Consider the operator equation

$$Au = f. \quad (1.2.27)$$

We assume the bilinear form $a(u, v) = \langle Au, v \rangle$ is bounded on $D(A) \times D(A)$:

$$|a(u, v)| \leq M|u||v|, \quad \forall u, v \in V,$$

which enables us to continuously extend $a(u, v)$ to $V \times V$. Thus we have obtained a variational, or a weak, form:

$$a(u, v) = \langle f, v \rangle, \quad \forall v \in V. \quad (1.2.28)$$

It is equivalent to an operator equation:

$$Tu = Rf. \quad (1.2.28')$$

A relationship between them is revealed by the Riesz representation theorem:

$$\begin{aligned} a(u, v) &= [Tu, v], \quad u, v \in V, \\ \langle f, v \rangle &= [Rf, v], \quad f \in V', v \in V. \end{aligned}$$

Apparently $T : V \rightarrow V$ is a bounded linear operator with $\|T\| \leq M$.

Let V be a separable Hilbert space. Then one can choose a sequence of finite dimensional subspaces $\{V_n\}$ of V such that

$$V_n \subset V_{n+1} \quad (n = 1, 2, \dots), \quad \overline{\bigcup_{n=1}^{\infty} V_n} = V.$$

Denote by Π_n the orthogonal projection operator from V to V_n . Then Π_n is a self-adjoint linear operator satisfying $\|\Pi_n\| = 1$ and

$$\lim_{n \rightarrow \infty} \|\Pi_n v - v\| = 0, \quad \forall v \in V.$$

The Galerkin method for the variational problem (1.2.28) is: Find $u_n \in V_n$ such that

$$a(u_n, v_n) = \langle f, v_n \rangle, \quad \forall v_n \in V_n, \quad (1.2.29)$$

or equivalently

$$T_n u_n = \Pi_n Rf \quad (T_n = \Pi_n T|_{V_n}). \quad (1.2.29')$$

Let $\{\phi_1, \dots, \phi_N\}$ be a basis of V_n . Write u_n as

$$u_n = \sum_{i=1}^N c_i \phi_i,$$

insert it into (1.2.29), and take $v_n = \phi_j$ ($1 \leq j \leq N$), then we have

$$\sum_{i=1}^N a(\phi_i, \phi_j) c_i = \langle f, \phi_j \rangle, \quad j = 1, 2, \dots, N. \quad (1.2.29'')$$

Solving it for c_i ($1 \leq i \leq N$) yields an approximate solution u_n . (1.2.29) (or an equivalent form) is called a Galerkin approximate equation, and u_n a Galerkin approximate solution.

Theorem 1.2.6 *Let V be a separable real Hilbert space, V_n a finite dimensional subspace of V , and $a(\cdot, \cdot)$ a bilinear form defined on $V \times V$ possessing the following properties:*

(i) *Boundedness: There is a constant $M > 0$ such that:*

$$|a(u, v)| \leq M|u||v|, \quad \forall u, v \in V;$$

(ii) *Positive definiteness: There exists a constant $\alpha > 0$ such that*

$$a(v, v) \geq \alpha|v|^2, \quad \forall v \in V.$$

Then, both the variational problem (1.2.28) and the approximation problem (1.2.29) have unique solutions u and u_n , respectively. Moreover, the following error estimate holds:

$$|u - u_n| \leq \frac{M}{\alpha} \inf_{v_n \in V_n} |u - v_n|. \quad (1.2.30)$$

Proof By Theorem 1.2.3, (1.2.28) has a unique solution. It follows from (ii) that the homogeneous equation

$$a(u_n, v_n) = 0, \quad \forall v_n \in V_n$$

corresponding to (1.2.29) admits only the trivial solution. Thus (1.2.29) has a unique solution for any given f . (1.2.28) and (1.2.29) lead to the error equation

$$a(u - u_n, v_n) = 0, \quad \forall v_n \in V_n. \quad (1.2.31)$$

So (1.2.30) follows from

$$\begin{aligned} |u - u_n|^2 &\leq \frac{1}{\alpha} a(u - u_n, u - u_n) \\ &= \frac{1}{\alpha} a(u - u_n, u - v_n) \leq \frac{M}{\alpha} |u - u_n| |u - v_n|, \quad \forall v_n \in V_n. \end{aligned}$$

This completes the proof. $\square$

Theorem 1.2.6 can also be deduced from Theorem 1.2.5 by noting that the positive definiteness implies (1.2.25).

(1.2.30) gives an error estimate of the approximate solution u_n in $|\cdot|$ norm, i.e., a V -estimate. Next we turn to discuss the error estimate of u_n in $\|\cdot\|$ norm, namely the H -estimate. To this end we use the Aubin-Nitsche dual argument.

Theorem 1.2.7 *Let the assumptions of Theorem 1.2.6 hold. Then*

$$\|u - u_n\| \leq M|u - u_n| \left(\sup_{g \in H} \frac{1}{\|g\|} \inf_{\tilde{\phi} \in V_n} |\phi - \tilde{\phi}| \right). \quad (1.2.32)$$

Here for any $g \in H$, $\phi \in V$ is the unique solution to the dual variational problem:

$$a(v, \phi) = (g, v), \quad \forall v \in V. \quad (1.2.33)$$

Proof The unique solvability of the dual problem (1.2.33) can be proved similarly to Theorem 1.2.6. To show (1.2.32) we use

$$\|u - u_n\| = \sup_{g \in H} \frac{|(g, u - u_n)|}{\|g\|}. \quad (1.2.34)$$

So setting $v = u - u_n$ in (1.2.33) yields

$$a(u - u_n, \phi) = (g, u - u_n).$$

By virtue of (1.2.31) we have

$$(g, u - u_n) = a(u - u_n, \phi - \tilde{\phi}), \quad \forall \tilde{\phi} \in V_n.$$

Now, employ the boundedness and take the infimum for $\tilde{\phi} \in V_n$ to get

$$|(g, u - u_n)| \leq M|u - u_n| \inf_{\tilde{\phi} \in V_n} |\phi - \tilde{\phi}|. \quad (1.2.35)$$

Finally, a combination of (1.2.34) and (1.2.35) leads to the desired result and completes the proof. $\square$

1.2.6 Generalized Galerkin methods

Let H be a separable real Hilbert space equipped with an inner product $(\cdot, \cdot)$ and the corresponding norm $\|\cdot\|$. U and V are dense linear subsets supplied with new inner products $[\cdot, \cdot]_U$, $[\cdot, \cdot]_V$ and related norms $|\cdot|_U$, $|\cdot|_V$, respectively. U and V respectively become Hilbert spaces under these new inner products. We also assume that the imbeddings of U and V in H are continuous.

Suppose A is a linear operator from a linear dense subset $D(A)$ of U to V' . For $f \in V'$ let us consider the operator equation:

$$Au = f. \quad (1.2.36)$$

Assume the bilinear form $a(u, v) = \langle Au, v \rangle$ on $D(A) \times V$ satisfies

$$|a(u, v)| \leq M|u|_U|v|_V, \quad \forall u \in D(A), v \in V. \quad (1.2.37)$$

Then as in Proposition 1.2.1 we may continuously extend $a(u, v)$ onto $U \times V$ such that the above estimate is still valid for any $(u, v) \in U \times V$. This results in a variational form of (1.2.36): Find $u \in U$ such that

$$a(u, v) = \langle f, v \rangle, \quad \forall v \in V. \quad (1.2.38)$$

It is equivalent to a bounded linear operator equation

$$Tu = Rf, \quad (1.2.38')$$

where $T : U \rightarrow V$ and $R : V' \rightarrow V$ are determined respectively by the Riesz representation theorem:

$$\begin{aligned} a(u, v) &= [Tu, v]_V, \quad u \in U, v \in V, \\ \langle f, v \rangle &= [Rf, v], \quad f \in V', v \in V. \end{aligned}$$

It is obvious that $\|T\| \leq M$.

The separability of the spaces enables us to choose two families of finite dimensional subspaces $\{U_n\}$ and $\{V_n\}$ of U and V respectively such that

$$U_n \subset U_{n+1}, \quad V_n \subset V_{n+1}, \quad n = 1, 2, \dots; \quad \overline{\bigcup_{n=1}^{\infty} U_n} = U, \quad \overline{\bigcup_{n=1}^{\infty} V_n} = V.$$

U_n is referred to as the trial function space, and V_n the test function space. The generalized Galerkin method for (1.2.38) is: Find $u_n \in U_n$ such that

$$a(u_n, v_n) = \langle f, v_n \rangle, \quad \forall v_n \in V_n, \quad (1.2.39)$$

or equivalently

$$T_n u_n = Q_n g, \quad (1.2.39')$$

where $T_n = Q_n T|_{U_n}$, $g = Rf$, and Q_n is the orthogonal projection operator from V to V_n .

Theorem 1.2.8 *Let U and V be separable real Hilbert spaces, U_n and V_n their subspaces respectively, and $a(\cdot, \cdot)$ the bilinear form defined on $U \times V$ satisfying*

$$|a(u, v)| \leq M|u|_U|v|_V, \quad \forall u \in U, v \in V, \quad (1.2.40)$$

$$\inf_{\substack{u_n \in U_n \\ |u_n|_U=1}} \sup_{\substack{v_n \in V_n \\ |v_n|_V=1}} |a(u_n, v_n)| \geq \alpha > 0, \quad (1.2.41)$$

where M and α are constants. Then, (1.2.38) and (1.2.39) possess unique solutions u and u_n respectively, and the following error estimate holds:

$$|u - u_n|_U \leq \left(1 + \frac{M}{\alpha}\right) \inf_{w \in U_n} |u - w|_U. \quad (1.2.42)$$

Proof Note that

$$|T_n u_n|_V = \sup_{\substack{v \in V \\ |v|_V=1}} |[T_n u_n, v]| \geq \sup_{\substack{v_n \in V_n \\ |v_n|_V=1}} |[T u_n, v_n]|.$$

So it follows from (1.2.41) that

$$|T_n u_n|_V \geq \alpha |u_n|_U, \quad \forall u_n \in U_n.$$

This together with Theorem 1.2.5 gives the desired result and completes the proof. $\square$

The estimate (1.2.42) in the above theorem indicates that the convergence order of $u - u_n$ is determined by the trial function space U_n , while the test function space V_n only influences the constant in the right-hand side of the estimate. This motivates us to speed up the convergence by choosing the trial function spaces with better approximate properties, and to simplify the approximation scheme by choosing simple and flexible test function spaces. It is this idea that the generalized difference method discussed in this book is based on.

Theorem 1.2.8 is difficult to apply in practice. So we shall modify the above framework to suit the need for the numerical analysis for the generalized difference method.

Let H be a separable real Hilbert space supplied with an inner product $(\cdot, \cdot)$ and the related norm $\|\cdot\|$. U is a dense linear subset of H , and is a Hilbert space with an inner product $[\cdot, \cdot]$ and the related norm $|\cdot|$. $a(\cdot, \cdot)$ is a bounded and positive definite bilinear form on $U \times U$. For $f \in H$, consider the equation

$$a(u, v) = (f, v), \quad \forall v \in U. \quad (1.2.43)$$

Set $D = \{u \in U : \text{the linear functional } a(u, \cdot) \text{ is continuous on } U \text{ with respect to the topology induced by } H\}$. In other words, D is such a subset of U that for each element u of D there exists a constant $M(u) > 0$ such that

$$|a(u, v)| \leq M(u)\|v\|, \quad \forall v \in U.$$

The density of U in H enables us to continuously extend $a(u, \cdot)$ to a bounded linear functional on H , and by the Riesz representation theorem there is a unique $Au \in H$ such that

$$a(u, v) = (Au, v), \quad u \in D, v \in H. \quad (1.2.44)$$

Obviously A is a linear operator from D to H .

Choose a trial function space $U_h \in U$ and a test function space $V_h \in H$ with dimensions $\dim U_h = \dim V_h = N$, where h is a parameter. Construct a discrete bilinear form $a_h(\cdot, \cdot)$ defined on $U_h \times V_h$ satisfying

$$a_h(u, v_h) = (Au, v_h), \quad \forall u \in D, v_h \in V_h. \quad (1.2.45)$$

The approximation scheme is: Find $u_h \in U_h$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \quad (1.2.46)$$

Let Γ_h be a linear operator from U to V_h satisfying $\Gamma_h U_h = V_h$. Then (1.2.46) is equivalent to

$$a_h(u_h, \Gamma_h w_h) = (f, \Gamma_h w_h), \quad \forall w_h \in U_h. \quad (1.2.46')$$

Theorem 1.2.9 Suppose $a_h(\cdot, \Gamma_h \cdot)$ is uniformly positive definite in the following sense: There exists a constant $\alpha > 0$ independent of the subspace U_h such that

$$a_h(w_h, \Gamma_h w_h) \geq \alpha |w_h|^2, \quad \forall w_h \in U_h. \quad (1.2.47)$$

Then (1.2.46) has a unique solution $u_h \in U_h$. If in addition the solution to (1.2.49) belongs to D , then we have the following error estimate:

$$|u - u_h| \leq \inf_{\tilde{u} \in U_h} \left\{ |u - \tilde{u}| + \frac{1}{\alpha} \sup_{w_h \in U_h} \frac{|a_h(u - \tilde{u}, \Gamma_h w_h)|}{|w_h|} \right\}. \quad (1.2.48)$$

Proof Consider the homogeneous equation related to (1.2.46)

$$a_h(u_h, v_h) = 0, \quad \forall v_h \in V_h.$$

Then we have $u_h = 0$ by setting $v_h = \Gamma_h u_h$ and using the uniformly positive definite condition (1.2.47). Thus the homogeneous equation admits only the trivial solution and consequently (1.2.46) has a unique solution.

Now let $u \in D$ and $u_h \in U_h$ be the solutions to (1.2.43) and (1.2.46) respectively. By (1.2.43)-(1.2.45) we know that

$$a_h(u, v_h) = (f, v_h), \quad \forall v_h \in V_h. \quad (1.2.49)$$

Subtracting (1.2.46) from (1.2.49) yields an error equation

$$a_h(u - u_h, v_h) = 0, \quad \forall v_h \in V_h. \quad (1.2.50)$$

For any $\tilde{u} \in U_h$, by (1.2.47) and (1.2.50) we have

$$\alpha |\tilde{u} - u_h|^2 \leq a_h(\tilde{u} - u_h, \Gamma_h(\tilde{u} - u_h)) = a_h(\tilde{u} - u, \Gamma_h(\tilde{u} - u_h)).$$

Hence

$$|\tilde{u} - u_h| \leq \frac{1}{\alpha} \sup_{w_h \in U_h} \frac{|a_h(\tilde{u} - u, \Gamma_h w_h)|}{|w_h|}.$$

Combine this equality with the triangular inequality

$$|u - u_h| \leq |u - \tilde{u}| + |\tilde{u} - u_h|,$$

and take the infimum with respect to $\tilde{u} \in U_h$, then we obtain (1.2.48). This completes the proof. $\square$

Bibliography and Comments

For the convenience of later use and the reader's reference, we provide in this chapter an outline of the Sobolev spaces and some basic results on their interpolation theories, variational problems and approximation methods. Most of the materials can be found in [A-27,26,19,2]. For a more systematical understanding of related topics, we refer to [B-1] for Sobolev space theory, [B-17] for the finite element method and the interpolation theory of Sobolev spaces, [A-26,3] and [B-47] for the generalized Galerkin method and the projection method. An original form of the general framework (Theorem 1.2.9) for the theory of generalized difference methods has been given in [A-30,53].

Chapter 2

TWO POINT BOUNDARY VALUE PROBLEMS

In this chapter we first illustrate the basic ideas of the generalized difference method by applying it to a second order ODE , i.e., a two point boundary value problem. It is shown how the generalized difference schemes are derived from the generalized Galerkin variational form. Then we present several examples of generalized difference schemes, and discuss their existence, uniqueness and convergence. Finally a fourth order problem is considered.

2.1 Basic Ideas of the Generalized Difference Method

2.1.1 A variational form

Consider the boundary value problem of the second order ODE on an interval $I = [a, b]$

$$(P1) \begin{cases} Lu \equiv -\frac{d}{dx}\left(p\frac{du}{dx}\right) + r\frac{du}{dx} + qu = f, & x \in (a, b), & (2.1.1a) \\ u(a) = 0, \quad u'(b) = 0, & & (2.1.1b) \end{cases}$$

where $p \in C^1(I)$, $p(x) \geq p_{\min} > 0$, and $r, q, f \in C(I)$.

Let $u \in C^1[a, b] \cap C^2(a, b)$ be the solution of (P1), and $H_E^1(I) = \{v \in H^1(I) : v(a) = 0\}$. Use any function $v \in H_E^1(I)$ (called a test function) to multiply (2.1.1a) and integrate it on $[a, b]$, then we have

$$\int_a^b Luv dx = \int_a^b f v dx.$$

By integrating by parts and the boundary condition (2.1.1b) we have

$$\begin{aligned} \int_a^b Luv dx &= \int_a^b (pu'v' + ru'v + quv) dx - pu'v|_a^b \\ &= \int_a^b (pu'v' + ru'v + quv) dx. \end{aligned}$$

Write

$$a(u, v) = \int_a^b (pu'v' + ru'v + quv) dx \quad (2.1.2)$$

and denote by $(\cdot, \cdot)$ the inner product of L^2 , then we find that u is the solution of the following problem:

$$(P2) \quad \begin{cases} \text{Find } u \in H_E^1(I) \text{ such that} \\ a(u, v) - (f, v) = 0, \forall v \in H_E^1(I). \end{cases} \quad (2.1.3)$$

On the other hand, if u is a solution of (P2) and $u \in C^1[a, b] \cap C^2(a, b)$, then integrating (2.1.3) by parts leads to

$$\int_a^b [-(pu')' + ru' + qu - f]v dx + p(b)u'(b)v(b) = 0, \forall v \in H_E^1(I). \quad (2.1.4)$$

In particular

$$\int_a^b (Lu - f)v dx = 0, \forall v \in C_0^\infty(I).$$

So u satisfies (2.1.1a) by the fundamental lemma of variational methods (Theorem 1.1.3). Now using (2.1.4) again we find that

$$p(b)u'(b)v(b) = 0, \forall v \in H_E^1(I).$$

Hence, setting $v(b) \neq 0$ shows $u'(b) = 0$. Therefore u satisfies (2.1.1b) as well and is the solution of (P1).

To sum up we have the following theorem.

Theorem 2.1.1 (Variational principle) *Suppose that $u \in C^1[a, b] \cap C^2(a, b)$ is the solution of (P1), then it is the solution of (P2). Conversely, if u is the solution of (P2) and $u \in C^1[a, b] \cap C^2(a, b)$, then it is the solution of (P1).*

(P1) is a differential form and (P2) is its Galerkin variational form. Solutions of (P1) are called classical solutions, while those of (P2) generalized solutions or weak solutions. In mechanics, the left hand side of (2.1.3) represents virtual work, and hence Theorem 2.1.1 is also referred to as a virtual work principle. There are significant differences between the two boundary conditions in (2.1.1b). The right boundary condition $u'(b) = 0$ needs not to be satisfied by the functions in $H_E^1(I)$. But it will be satisfied *naturally* by the solution of the variational problem. Therefore, it is called a natural boundary condition. In mechanics, this boundary condition corresponds to the force. On the contrary, the left boundary condition $u(a) = 0$ must be imposed on $H_E^1(I)$. Hence it is called an essential boundary condition. It is a geometrical condition. Variational principles are commonly used to describe physical phenomena and also lay the foundation of the numerical methods for differential equations. Compared with the differential forms, the merit of the variational problems is that, for instance, the second derivative of u is not involved in the problem (P2), and the natural boundary conditions are much easier to deal with.

2.1.2 Galerkin methods

It is usually difficult to directly solve the variational forms to get the precise solutions. The main trouble lies in the fact that $H_E^1(I)$ is an infinite dimensional space. The idea to overcome this is to approximate infinite dimensional spaces by finite ones.

Let us choose a finite dimensional subspace U_h in $U = H_E^1(I)$, and use it to replace U in (P2) to obtain an approximate problem:

$$(P2)_h \begin{cases} \text{Find } u_h \in U_h \text{ such that} \\ a(u_h, v_h) = (f, v_h), \forall v_h \in U_h. \end{cases} \quad (2.1.5)$$

This is precisely the so-called Galerkin method, or the variational method, and u_h is the Galerkin approximation of u .

Galerkin methods in the early stages use smooth functions (usually algebraic or triangular polynomials, or special functions related to certain specific problems) to construct the finite dimensional space U_h . There are some disadvantages to follow this approach in practice. Mainly they are: the difficulties in constructing globally defined polynomials to satisfy the boundary conditions for multidimensional irregular regions; the big computing work for calculating the integrals to form the Galerkin equation; and the non-sparseness and the large condition number of the coefficient matrix of the Galerkin equation. Therefore the classical Galerkin methods cannot match finite difference methods which possess, on the contrary, advantages such as sparse coefficient matrices, less computing work and simple programming. The finite element method, initiated by R. Courant (1943) and developed in the fifties, provides a new approach for Galerkin methods to construct the subspace U_h . It decomposes the solution region into a network like the difference method, and utilizes spline functions to construct the subspace U_h , which contains low order piecewise polynomials satisfying the essential boundary conditions as well as certain global smoothness. Such kinds of Galerkin methods are called finite element methods or Galerkin finite element methods, and U_h 's the finite element spaces. The coefficient matrices of the finite element method are sparse, and their computation is simple and flexible. It is particularly powerful in dealing with irregular regions and natural boundary conditions. Therefore, the finite element method is an effective numerical method for elliptic and parabolic equations.

2.1.3 Generalized Galerkin variational principles

Our generalized difference methods are based on the traditional difference methods and have absorbed certain ideas of finite element methods (mainly the variational forms and the finite element spaces). We will see that the generalized difference methods enjoy the advantages of both the difference methods and the finite element methods. To set up the generalized difference method and its theoretical foun-

dation, we need to design a new variational principle of generalized Galerkin type. In this chapter we only take a one-dimensional problem as an example to illustrate the idea.

Let us discretize the interval $I = [a, b]$ into a set of points (or nodes)

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

The subintervals $I_i = [x_{i-1}, x_i]$ are called elements. All these elements compose a discretization of I , denoted by $\sigma = \{I_i : 1 \leq i \leq n\}$. Denote by I_i^0 the interior of I_i , and $\mathcal{P}_k$ the set of all the polynomials of degrees less than or equal to k . Write

$$S_\sigma^{(k)}(I) = \{v \in L^2(I) : v|_{I_i^0} \in \mathcal{P}_k, i = 1, 2, \dots, n\}$$

and call it the set of piecewise polynomials of degree k with respect to σ . Similarly call

$$S^{(k)}(I) = \bigcup_{\sigma} S_\sigma^{(k)}(I)$$

the piecewise polynomials of degree k on I . In particular, it is called the set of piecewise constant (or step) and piecewise linear functions when $k = 0$ and $k = 1$ respectively. We also write

$$S_{\sigma, E}^{(k)}(I) = \{v \in S_\sigma^{(k)}(I) : v(a^+) = 0\},$$

$$S_E^{(k)}(I) = \bigcup_{\sigma} S_{\sigma, E}^{(k)}(I).$$

Use any $v \in V = S_E^{(k)}(I)$ to multiply equation (2.1.1a), and integrate it on I . Then by integrating by parts we have

$$(Lu, v) = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (pu'v' + ru'v + quv)dx - \sum_{i=1}^n pu'v \Big|_{x_{i-1}^+}^{x_i^-}. \quad (2.1.6)$$

Noticing $v(a^+) = 0$ and $u'(b) = 0$, we know that u satisfies

$$a_\sigma(u, v) = (f, v), \quad \forall v \in V, \quad (2.1.7)$$

where

$$a_\sigma(u, v) = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (pu'v' + ru'v + quv)dx + \sum_{i=1}^{n-1} p(x_i)u'(x_i)[v(x_i^+) - v(x_i^-)]. \quad (2.1.8)$$

Let us introduce a generalized function $\delta(x)$ defined as the derivative of the following jump function:

$$\sigma(x) = \begin{cases} 0, & \text{for } x < 0, \\ 1, & \text{for } x > 0. \end{cases}$$

So formally we have

$$\delta(x) = \sigma'(x) = \begin{cases} 0, & \text{for } x \neq 0, \\ \infty, & \text{for } x = 0. \end{cases}$$

For any smooth function $g(x)$ we have

$$\int_\alpha^\beta g(x)\delta(x)dx = g(0), \quad \alpha < 0 < \beta.$$

The piecewise polynomial function $v \in V$ mentioned above can be expressed as the sum of a continuous function v_1 and a step function v_2 :

$$v = v_1 + v_2, \\ v_2 = \sum_{i=1}^{n-1} [v(x_i^+) - v(x_i^-)]\sigma(x - x_i).$$

So if u' is continuous or $u \in H^2(I)$, then in the sense of generalized functions we have

$$\begin{aligned} a(u, v) &\equiv \int_a^b (pu'v' + ru'v + quv)dx \\ &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (pu'v' + ru'v + quv)dx \\ &\quad + \sum_{i=1}^{n-1} p(x_i)u'(x_i)[v(x_i^+) - v(x_i^-)] \\ &= a_\sigma(u, v). \end{aligned} \quad (2.1.9)$$

Now, it follows from (2.1.7) and (2.1.9) that the solution u of (P1) also solves the following problem

$$(P3) \quad \begin{cases} \text{Find } u \in H_E^1(I) \cap H^2(I) \text{ such that} \\ a(u, v) - (f, v) = 0, \forall v \in V. \end{cases} \quad (2.1.10)$$

On the other hand, if u is a solution of (P3) and $u \in C^1[a, b] \cap C^2(a, b)$, then it follows from (2.1.6) and (2.1.7) that

$$(Lu - f, v) + p(b)u'(b)v(b^-) = 0, \forall v \in V. \quad (2.1.11)$$

In particular

$$(Lu - f, v) = 0, \forall v \in \{v \in V : v(b^-) = 0\}.$$

Since the set $\{v \in V : v(b^-) = 0\}$ is dense in $L^2(I)$, the above equation is valid for any $v \in L^2(I)$. This verifies (2.1.1a). Now (2.1.11) becomes

$$p(b)u'(b)v(b^-) = 0, \forall v \in V.$$

Taking $v \in V$ with $v(b^-) \neq 0$ implies $u'(b) = 0$. Finally, we note $u \in H_E^1(I)$, so (2.1.1b) is valid and u is the solution of (P1).

The above discussion leads to the following theorem.

Theorem 2.1.2 *Suppose that $u \in C^1[a, b] \cap C^2(a, b)$ is the solution of (P1); then it is the solution of (P3). Conversely, if u is the solution of (P3) and $u \in C^1[a, b] \cap C^2(a, b)$, then it is the solution of (P1).*

We shall call (P3) the generalized Galerkin variational problem with respect to (P1), and Theorem 2.1.2 the generalized Galerkin variational principle.

Now we make a convention that in the sequel the bilinear form $a(u, v)$ is understood according to (2.1.9), which coincides with the original definition when $v \in H_E^1(I)$, and that (2.1.2) should be interpreted in terms of generalized functions when $v \in S_E^{(k)}(I)$.

2.1.4 Generalized difference methods

Let us decompose the interval $I = [a, b]$ into a grid T_h with nodes

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Accordingly we place a dual grid

$$a = x_0 < x_{1/2} < x_{3/2} < \cdots < x_{n-1/2} < x_n = b,$$

where $x_{i-1/2} = (x_{i-1} + x_i)/2$, $i = 1, 2, \dots, n$. Write $I_0^* = [x_0, x_{1/2}]$, $I_i^* = [x_{i-1/2}, x_{i+1/2}]$ ($i = 1, 2, \dots, n-1$) and $I_n^* = [x_{n-1/2}, x_n]$. Then, these dual elements I_i^* ($i = 0, 1, 2, \dots, n$) lead to a dual grid, written as T_h^* .

Choose the trial function space $U_h \subset U = H_E^1(I)$ as a finite element space with respect to T_h , and the test function space $V_h \subset V = S_E^{(k)}(I)$ as a piecewise polynomial (of low order) space $S_{\sigma^*, E}^{(k)}$ with respect to the dual discretization σ^* induced by T_h^* . Now we propose an approximation problem of (P3)

$$(P3)_h \begin{cases} \text{Find } u_h \in U_h \text{ such that} \\ a(u_h, v_h) = (f, v_h), \forall v_h \in V_h. \end{cases} \quad (2.1.12)$$

Different choices of U_h and V_h lead to different schemes. In particular, selecting U_h and V_h as piecewise linear and constant functions respectively yields the usual difference scheme, as we shall see in the next section. This explains why we call $(P3)_h$ the generalized difference method.

Let us elaborate on the general considerations in constructing the test function space V_h . The choice of V_h certainly should be somehow related to U_h . For instance, V_h should have the same degree of freedom as U_h . But it is not required for the functions in V_h to possess global continuities, so we can choose V_h as low order piecewise polynomial spaces to reduce the computation effort. Usually when the values $d^j u(x_i)/dx^j$ ($j \geq 0$) make sense at the node x_i for the functions in U_h , one can choose the basis function $\psi_i^{(j)}$ of V_h with respect to the point x_i to satisfy the following conditions:

- (i) The support of $\psi_i^{(j)}$ belongs to the dual element I_i^* containing x_i .

(ii) On I_i^*

$$\left. \frac{d^l}{dx^l} \psi_i^{(j)}(x) \right|_{x=x_i} = \begin{cases} 0, & \text{for } l \neq j, \\ 1, & \text{for } l = j. \end{cases}$$

These conditions imply that

$$\psi_i^{(j)}(x) = \begin{cases} (x - x_i)^j / j!, & x \in I_i^*, \\ 0, & x \notin I_i^*. \end{cases}$$

Finally, we point out that the generalized difference methods are different from the usual generalized Galerkin finite element methods or the nonstandard finite element methods, in that we only have $V_h \in L^2$ rather than $V_h \in H^1(I)$ and the corresponding bilinear form has to be understood in the sense of generalized functions. So the variational forms of the generalized difference methods differ essentially from the usual ones, causing difficulties in theoretical analysis.

2.2 Linear Element Difference Schemes

Consider the two point boundary value problem

$$\begin{cases} Lu \equiv -\frac{d}{dx} \left(p \frac{du}{dx} \right) = f, & x \in (a, b), \\ u(a) = 0, & u'(b) = 0, \end{cases} \quad \begin{matrix} (2.2.1a) \\ (2.2.1b) \end{matrix}$$

where $p \in C^1(I)$, $p(x) \geq p_{\min} > 0$, and $f \in L^2(I)$.

In this section we deduce a linear element difference scheme by choosing the trial and the test function spaces as linear finite element and piecewise constant function spaces respectively. This will result in the usual difference scheme as we have promised.

2.2.1 Trial and test function spaces

Discretize the interval $I = [a, b]$ into a grid T_h with nodes

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Denote the length of the element I_i by $h_i = x_i - x_{i-1}$ and write $h = \max_{1 \leq i \leq n} h_i$. We assume the grid satisfies the quasi-uniform condition $h_i \geq \mu h$ ($i = 1, 2, \dots, n$) for some positive constant μ .

The trial space U_h is taken as the linear element space with respect to T_h , which consists of all the functions u_h satisfying

(i) $u_h \in C(I)$, $u_h(a) = 0$ and

(ii) u_h is linear on each I_i and is determined uniquely by its values at the endpoints of the element.

Obviously U_h is an n -dimensional subspace of $H_E^1(I)$.

To construct the nodal basis functions we consider an interpolation problem on the reference element $[0,1]$: Find a linear function $N_0(\xi)$ such that

$$N_0(0) = 1, \quad N_0(1) = 0.$$

Then, $N_0(\xi) = 1 - \xi$. Notice that the affine mapping

$$\xi = \frac{x_i - x}{h_i}$$

maps the left interval $I_i = [x_{i-1}, x_i]$ of the node x_i onto the reference element $[0,1]$ with $x_i \rightarrow 0$, $x_{i-1} \rightarrow 1$. Thus, on the interval $[x_{i-1}, x_i]$ the basis function ϕ_i of the node x_i is of the form

$$\phi_i(x) = 1 - \frac{x_i - x}{h_i}.$$

Similarly the affine mapping

$$\xi = \frac{x - x_i}{h_{i+1}}$$

maps the right interval $I_i = [x_i, x_{i+1}]$ of the node x_i onto the reference element $[0,1]$ with $x_i \rightarrow 0$, $x_{i+1} \rightarrow 1$. Thus, on the interval $[x_i, x_{i+1}]$ ϕ_i can be expressed as

$$\phi_i(x) = 1 - \frac{x - x_i}{h_{i+1}}.$$

Therefore, the basis function with respect to x_i is

$$\phi_i(x) = \begin{cases} 1 - h_i^{-1}|x - x_i|, & x_{i-1} \leq x \leq x_i, \\ 1 - h_{i+1}^{-1}|x - x_i|, & x_i \leq x \leq x_{i+1}, \\ 0, & \text{elsewhere.} \end{cases} \quad (2.2.2)$$

The functions $\{\phi_i(x) : i = 1, 2, \dots, n\}$ form a basis of U_h and any $u_h \in U_h$ has the following expression

$$u_h = \sum_{i=1}^n u_i \phi_i(x),$$

where $u_i = u_h(x_i)$. On the element I_i we have

$$u_h = u_{i-1}(1 - \xi) + u_i \xi, \quad \left(\xi = \frac{x - x_{i-1}}{h_i} \right) \quad (2.2.3)$$

$$u'_h = (u_i - u_{i-1})/h_i, \quad x \in I_i, \quad i = 1, 2, \dots, n. \quad (2.2.4)$$

Next we place a dual grid T_h^* with nodes

$$a = x_0 < x_{1/2} < x_{3/2} < \dots < x_{n-1/2} < x_n = b,$$

where $x_{i-1/2} = (x_{i-1} + x_i)/2$, $i = 1, 2, \dots, n$. The dual elements are $I_0^* = [x_0, x_{1/2}]$, $I_i^* = [x_{i-1/2}, x_{i+1/2}]$ ($i = 1, 2, \dots, n-1$) and $I_n^* = [x_{n-1/2}, x_n]$. Accordingly we choose the test function space V_h as the piecewise constant function (step function) space, which contains all the functions $v_h \in L^2(I)$ satisfying

- (i) $v_h(x) = 0$, for $x \in I_0^*$ and
- (ii) v_h is a constant on each I_i^* ($i = 1, 2, \dots, n$).

The basis functions of V_h are

$$\psi_j(x) = \begin{cases} 1, & x \in I_j^*, \\ 0, & x \notin I_j^*, \end{cases} \quad j = 1, 2, \dots, n. \quad (2.2.5)$$

Any $v_h \in V_h$ has the form

$$v_h = \sum_{i=1}^n v_i \psi_i(x),$$

where $v_i = v_h(x_i)$.

Typical basis functions of U_h and V_h are depicted in Fig. 2.2.1.

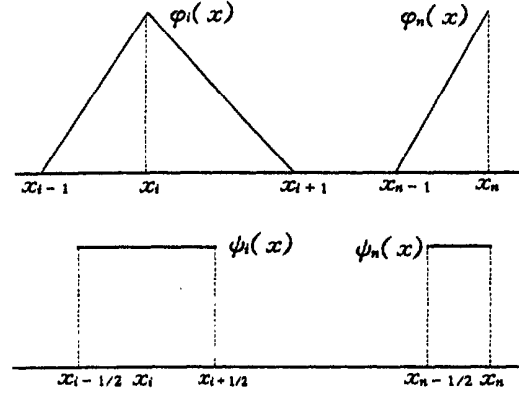


Fig. 2.2.1

2.2.2 Difference equations

Following (P3)_h in Subsection 2.1.4, the linear element difference scheme is: Find $u_h = \sum_{i=1}^n u_i \phi_i(x)$ such that

$$a(u_h, \psi_j) = (f, \psi_j), \quad j = 1, 2, \dots, n, \quad (2.2.6)$$

where (cf. (2.1.9))

$$\begin{aligned} a(u_h, \psi_j) &= \int_a^b p u_h' [\delta(x - x_{j-1/2}) - \delta(x - x_{j+1/2})] dx \\ &= p_{j-1/2} u_h'(x_{j-1/2}) - p_{j+1/2} u_h'(x_{j+1/2}) \\ &= p_{j-1/2} (u_j - u_{j-1}) / h_j - p_{j+1/2} (u_{j+1} - u_j) / h_{j+1}, \\ &\quad j = 1, 2, \dots, n-1, \\ &\quad u_0 = 0, \\ a(u_h, \psi_n) &= p_{n-1/2} (u_n - u_{n-1}) / h_n. \end{aligned}$$

This results in a difference equation

$$\begin{cases} p_{j-1/2}(u_j - u_{j-1})/h_j - p_{j+1/2}(u_{j+1} - u_j)/h_{j+1} = \int_{x_{j-1/2}}^{x_{j+1/2}} f dx, \\ j = 1, 2, \dots, n-1, \quad u_0 = 0, \\ p_{n-1/2}(u_n - u_{n-1})/h_n = \int_{x_{n-1/2}}^{x_n} f dx. \end{cases} \quad (2.2.7)$$

The left-hand side of this equation coincides with the usual finite difference method. Furthermore, if we use $f_j = f(x_j)$ and $f_n = f(x_n)$ respectively to approximate the integrals in (2.2.7), then we end up with precisely the usual finite difference scheme. Therefore, we see that a finite difference scheme has been derived from the generalized difference method.

(2.2.7) is a linear system for the unknowns $u_1, u_2, \dots, u_n$ with a symmetric tridiagonal coefficient matrix:

$$\begin{bmatrix} \frac{p_{\frac{1}{2}}}{h_1} + \frac{p_{\frac{3}{2}}}{h_2} & -\frac{p_{\frac{3}{2}}}{h_2} & & & \\ -\frac{p_{\frac{3}{2}}}{h_2} & \frac{p_{\frac{3}{2}}}{h_2} + \frac{p_{\frac{5}{2}}}{h_3} & -\frac{p_{\frac{5}{2}}}{h_3} & & \\ & -\frac{p_{\frac{5}{2}}}{h_3} & \frac{p_{\frac{5}{2}}}{h_3} + \frac{p_{\frac{7}{2}}}{h_4} & -\frac{p_{\frac{7}{2}}}{h_4} & \\ & & \ddots & \ddots & \ddots \\ & & & -\frac{p_{n-\frac{3}{2}}}{h_{n-1}} & \frac{p_{n-\frac{3}{2}}}{h_{n-1}} + \frac{p_{n-\frac{1}{2}}}{h_n} & -\frac{p_{n-\frac{1}{2}}}{h_n} \\ & & & & -\frac{p_{n-\frac{1}{2}}}{h_n} & \frac{p_{n-\frac{1}{2}}}{h_n} \end{bmatrix}$$

2.2.3 Convergence estimates

There have been thorough discussions for the convergence of the finite difference scheme corresponding to (2.2.6) (or (2.2.7)). Here we consider the error estimates of (2.2.6) in a Sobolev norm.

First, for $u_h \in U_h$ we have by (2.2.4) that

$$|u_h|_1 = \left[\int_a^b (u_h')^2 dx \right]^{1/2} = \left[\sum_{i=1}^n (u_i - u_{i-1})^2 / h_i \right]^{1/2}. \quad (2.2.8)$$

Next, we define an interpolation operator $\Pi_h^* : U \rightarrow V_h$ by

$$\Pi_h^* u = \sum_{j=1}^n u_j \psi_j, \quad \forall u \in U.$$

Finally we examine the positive definiteness of $a(u_h, \Pi_h^* u_h)$.

$$\begin{aligned} a(u_h, \Pi_h^* u_h) &= \sum_{j=1}^n u_j a(u_h, \psi_j) \\ &= \sum_{j=1}^{n-1} u_j [p_{j-1/2}(u_j - u_{j-1})/h_j - p_{j+1/2}(u_{j+1} - u_j)/h_{j+1}] \\ &\quad + u_n p_{n-1/2}(u_n - u_{n-1})/h_n \\ &= \sum_{j=1}^n p_{j-1/2}(u_j - u_{j-1})^2 / h_j \geq p_{\min} |u_h|_1^2. \end{aligned}$$

This gives the following theorem.

Theorem 2.2.1 *The discrete bilinear form $a(u_h, \Pi_h^* u_h)$ is positive definite, that is, there exists a constant $\alpha > 0$ such that*

$$a(u_h, \Pi_h^* u_h) \geq \alpha |u_h|_1^2, \quad \forall u_h \in U_h. \quad (2.2.9)$$

We notice that the seminorm $|\cdot|_1$ and the norm $\|\cdot\|_1$ are equivalent in the space $H_E^1(I)$. Thus the existence and uniqueness of the difference scheme (2.2.6) follows from Theorem 2.2.1. Now we turn to the convergence estimate.

Theorem 2.2.2 *Let $u \in H^2(I)$ be the solution of the differential equation (2.2.1) and u_h the solution of the difference scheme (2.2.6), then the following estimate holds:*

$$|u - u_h|_1 \leq Ch |u|_2. \quad (2.2.10)$$

Proof It is obvious that

$$a(u, \psi_j) = (f, \psi_j), \quad j = 1, 2, \dots, n,$$

$$a(u_h, \psi_j) = (f, \psi_j), \quad j = 1, 2, \dots, n.$$

So

$$a(u - u_h, \psi_j) = 0, \quad j = 1, 2, \dots, n. \quad (2.2.11)$$

Let $\Pi_h u$ be the interpolation projection of u onto the trial function space U_h . Then, by Theorem 2.2.1 and (2.2.11) we get

$$\begin{aligned} |u_h - \Pi_h u|_1^2 &\leq \frac{1}{\alpha} a(u_h - \Pi_h u, \Pi_h^*(u_h - \Pi_h u)) \\ &= \frac{1}{\alpha} a(u - \Pi_h u, \Pi_h^*(u_h - \Pi_h u)). \end{aligned}$$

Thus

$$|u_h - \Pi_h u|_1 \leq \frac{1}{\alpha} \sup_{w_h \in U_h} \frac{|a(u - \Pi_h u, \Pi_h^* w_h)|}{|w_h|_1}. \quad (2.2.12)$$

Write $w_j = w_h(x_j)$, then we have $\Pi_h^* w_h = \sum_{j=1}^n w_j \psi_j$ and

$$\begin{aligned} a(u - \Pi_h u, \Pi_h^* w_h) &= \sum_{j=1}^n w_j a(u - \Pi_h u, \psi_j) \\ &= \sum_{j=1}^{n-1} w_j [p_{j-1/2}(u - \Pi_h u)'_{j-1/2} - p_{j+1/2}(u - \Pi_h u)'_{j+1/2} \\ &\quad + w_n p_{n-1/2}(u - \Pi_h u)'_{n-1/2}] \\ &= \sum_{j=1}^{n-1} p_{j-1/2}(u - \Pi_h u)'_{j-1/2} (w_j - w_{j-1}). \end{aligned}$$

By the Cauchy inequality we have

$$\begin{aligned} &|a(u - \Pi_h u, \Pi_h^* w_h)| \\ &\leq C \left\{ \sum_{j=1}^n [(u - \Pi_h u)'_{j-1/2}]^2 \right\}^{1/2} \left\{ \sum_{j=1}^n (w_j - w_{j-1})^2 \right\}^{1/2}. \end{aligned} \quad (2.2.13)$$

By (2.2.4)

$$(u - \Pi_h u)'_{j-1/2} = u'_{j-1/2} - (u_j - u_{j-1})/h_j.$$

By the mean value theorem there exists $\xi_0 \in I_j$ such that

$$u'(\xi_0) = (u_j - u_{j-1})/h_j, \text{ i.e. } (u - \Pi_h u)'(\xi_0) = 0.$$

Hence,

$$(u - \Pi_h u)'_{j-1/2} = \int_{\xi_0}^{x_{j-1/2}} (u - \Pi_h u)'' dx = \int_{\xi_0}^{x_{j-1/2}} u'' dx,$$

which yields

$$|(u - \Pi_h u)'_{j-1/2}|^2 \leq h \left[\int_{x_{j-1}}^{x_j} (u'')^2 dx \right],$$

$$\left\{ \sum_{j=1}^n [(u - \Pi_h u)'_{j-1/2}]^2 \right\}^{1/2} \leq h^{1/2} |u|_2. \quad (2.2.14)$$

Combining (2.2.13), (2.2.14) and (2.2.8) we get

$$|a(u - \Pi_h u, \Pi_h^* w_h)| \leq Ch |u|_2 |w_h|_1.$$

This together with (2.2.12) yields

$$|u_h - \Pi_h u|_1 \leq Ch |u|_2.$$

By the interpolation theory in Sobolev spaces

$$|u - \Pi_h u|_1 \leq Ch |u|_2.$$

These two estimates imply (2.2.10) and complete the proof. $\square$

2.3 Quadratic Element Difference Schemes

Consider the two point boundary value problem:

$$\begin{cases} Lu \equiv -\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu = f, & x \in (a, b), \\ u(a) = 0, \quad u'(b) = 0, \end{cases} \quad \begin{matrix} (2.3.1a) \\ (2.3.1b) \end{matrix}$$

where $p, q \in C^1(I)$, $p(x) \geq p_{\min} > 0$, $q \geq 0$ and $f \in L^2(I)$.

In this section we derive a quadratic element difference scheme by choosing the trial and the test spaces as the quadratic element space of Lagrangian type and the piecewise constant function space respectively.

2.3.1 Trial and test spaces

Perform the same discretization as in §2.2 to get the grid T_h . But now the interpolation points include, besides the nodal points x_j , the midpoints of the elements $x_{j-1/2} = (x_j + x_{j-1})/2$ ($j = 1, 2, \dots, n$) as well.

Select the trial function space U_h as the quadratic element space of Lagrangian type with respect to T_h . So any function u_h in U_h satisfies the following conditions:

- (i) $u_h \in C(I)$, $u_h(a) = 0$;
- (ii) u_h is a quadratic polynomial at each element I_i and it is determined uniquely by its values at the two endpoints and the midpoint of the element.

Obviously U_h is a $2n$ -dimensional subspace of $U = H_E^1(I)$.

To obtain the basis functions, let us first construct the quadratic functions $N_0(\xi)$ and $N_{1/2}(\xi)$ such that

$$\begin{aligned} N_0(0) &= 1, & N_0(0.5) &= N_0(1) = 0, \\ N_{1/2}(0.5) &= 1, & N_{1/2}(0) &= N_{1/2}(1) = 0. \end{aligned}$$

It is an easy matter to get

$$\begin{aligned} N_0(\xi) &= (2\xi - 1)(\xi - 1), \\ N_{1/2}(\xi) &= 4\xi(1 - \xi). \end{aligned}$$

Setting $\xi = (x - x_i)/h_{i+1}$ and $\xi = (x_i - x)/h_i$ respectively in $N_0(\xi)$, we end up with the right and left halves of the basis function with respect to the node x_i . So we have

$$\phi_i(x) = \begin{cases} (2|x - x_i|/h_i - 1)(|x - x_i|/h_i - 1), & x_{i-1} \leq x \leq x_i, \\ (2|x - x_i|/h_{i+1} - 1)(|x - x_i|/h_{i+1} - 1), & x_i \leq x \leq x_{i+1}, \\ 0, & \text{elsewhere.} \end{cases}$$

Similarly set $\xi = (x - x_{j-1})/h_i$ in $N_{1/2}(\xi)$ to get the basis function with respect to the node $x_{i-1/2}$

$$\phi_{i-1/2}(x) = \begin{cases} 4(1 - (x - x_{i-1})/h_i)(x - x_{i-1})/h_i, & x_{i-1} \leq x \leq x_i, \\ 0, & \text{elsewhere.} \end{cases}$$

The set $\{\phi_i(x), \phi_{i-1/2}(x); 1 \leq i \leq n\}$ is a basis of U_h and any $u_h \in U_h$ can be uniquely written as

$$u_h = \sum_{i=1}^n [u_i \phi_i(x) + u_{i-1/2} \phi_{i-1/2}(x)],$$

where $u_i = u_h(x_i)$ and $u_{i-1/2} = u_h(x_{i-1/2})$. In the element $I_i = [x_{i-1}, x_i]$

$$\begin{aligned} u_h &= u_{i-1}(2\xi - 1)(\xi - 1) + 4u_{i-1/2}\xi(1 - \xi) + u_i(2\xi - 1)\xi \\ &= (\xi^2, \xi, 1) \begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{i-1} \\ u_{i-1/2} \\ u_i \end{bmatrix}, \end{aligned} \tag{2.3.2}$$

$$\begin{aligned} u'_h &= u_{i-1}(4\xi - 3)/h_i + u_{i-1/2}(-8\xi + 4)/h_i + u_i(4\xi - 1)/h_i \\ &= (\xi, 1) \begin{bmatrix} -4 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} (u_{i-1/2} - u_{i-1})/h_i \\ (u_i - u_{i-1/2})/h_i \end{bmatrix}, \end{aligned} \tag{2.3.3}$$

where $\xi = (x - x_{i-1})/h_i$.

Next, we place a dual grid T_h^* with nodal points

$$a = x_0 < x_{1/4} < x_{3/4} < \cdots < x_{n-3/4} < x_{n-1/4} < x_n = b,$$

where $x_{i-k/4} = x_i - \frac{k}{4}h_i$ ($k = 1, 3$, $i = 1, 2, \dots, n$). The test function space corresponding to T_h^* is taken as the piecewise constant function space, which is a $2n$ -dimensional subspace spanned by the basis functions of the nodes x_i

$$\psi_i(x) = \begin{cases} 1, & x_{i-1/4} \leq x \leq x_{i+1/4}, \\ 0, & \text{elsewhere,} \end{cases}$$

and the ones of $x_{i-1/2}$

$$\psi_{i-1/2}(x) = \begin{cases} 1, & x_{i-3/4} \leq x \leq x_{i-1/4}, \\ 0, & \text{elsewhere.} \end{cases}$$

Any $v_h \in V_h$ can be uniquely expressed as

$$v_h = \sum_{j=1}^n [v_j \psi_j(x) + v_{j-1/2} \psi_{j-1/2}(x)].$$

Typical basis functions ϕ_i of U_h and ψ_i of V_h are depicted in Fig. 2.3.1.

2.3.2 Difference equations

The quadratic difference scheme corresponding to the subspaces U_h and V_h given in the last subsection is: Find $u_h \in U_h$ such that

$$\begin{cases} a(u_h, \psi_j) = (f, \psi_j), & j = 1, 2, \dots, n, \\ a(u_h, \psi_{j-1/2}) = (f, \psi_{j-1/2}), & j = 1, 2, \dots, n, \end{cases} \quad (2.3.4)$$

where

$$\begin{aligned} a(u_h, \psi_j) &= p_{j-1/4} u_h'(x_{j-1/4}) - p_{j+1/4} u_h'(x_{j+1/4}) + \int_{x_{j-1/4}}^{x_{j+1/4}} q u_h dx \\ &= 2p_{j-1/4} (u_j - u_{j-1/2})/h_j - 2p_{j+1/4} (u_{j+1/2} - u_j)/h_{j+1} \\ &\quad + \int_{x_{j-1/4}}^{x_{j+1/4}} q u_h dx, \end{aligned} \quad (2.3.5)$$

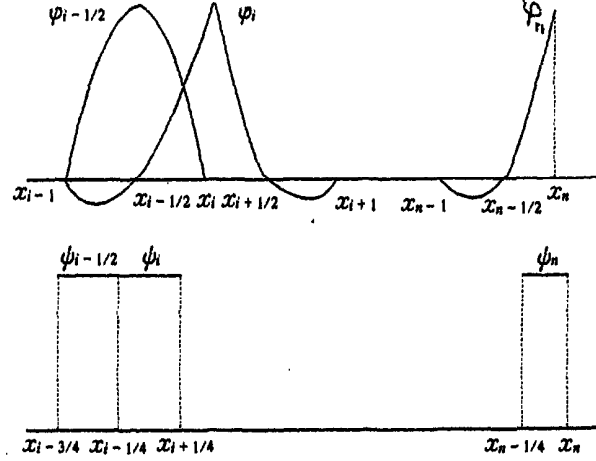


Fig. 2.3.1

$$\begin{aligned}
 & a(u_h, \psi_{j-1/2}) \\
 &= p_{j-3/4} u'_h(x_{j-3/4}) - p_{j-1/4} u'_h(x_{j-1/4}) + \int_{x_{j-3/4}}^{x_{j-1/4}} q u_h dx \\
 &= 2p_{j-3/4}(u_{j-1/2} - u_{j-1})/h_j - 2p_{j-1/4}(u_j - u_{j-1/2})/h_j \\
 &\quad + \int_{x_{j-3/4}}^{x_{j-1/4}} q u_h dx.
 \end{aligned} \tag{2.3.6}$$

In the above expressions $u_0 = 0$ and, when $j = n$, the quantities on the right-hand side of $x_n = b$ should be dropped. For this reason we make the convention that $p_{n+1/4} = 0$ and $x_{n+1/4} = x_n$.

Exploiting the numerical quadrature formula

$$\begin{aligned}
 \int_{x_{j-1/4}}^{x_{j+1/4}} q u_h dx &= \frac{1}{4}(h_j + h_{j+1})q_j u_j \\
 \int_{x_{j-3/4}}^{x_{j-1/4}} q u_h dx &= \frac{1}{2}h_j q_{j-1/2} u_{j-1/2}
 \end{aligned}$$

leads to the following difference equation corresponding to x_j

$$\begin{aligned}
& a_h(u_h, \psi_j) \\
\equiv & p_{j-1/4} \frac{2(u_j - u_{j-1/2})}{h_j} - p_{j+1/4} \frac{2(u_{j+1/2} - u_j)}{h_{j+1}} + \frac{h_j + h_{j+1}}{4} q_j u_j \\
& = \int_{x_{j-1/4}}^{x_{j+1/4}} f dx, \quad j = 1, 2, \dots, n, \quad (2.3.7)
\end{aligned}$$

and the one to $x_{j-1/2}$

$$\begin{aligned}
& a_h(u_h, \psi_{j-1/2}) \\
\equiv & p_{j-3/4} \frac{2(u_{j-1/2} - u_{j-1})}{h_j} - p_{j-1/4} \frac{2(u_j - u_{j-1/2})}{h_j} \\
& + \frac{h_j}{2} q_{j-1/2} u_{j-1/2} \\
= & \int_{x_{j-3/4}}^{x_{j-1/4}} f dx, \quad j = 1, 2, \dots, n. \quad (2.3.8)
\end{aligned}$$

This gives a finite difference scheme on the given grid. If the unknowns are arranged in the order $u_{1/2}, u_1, u_{3/2}, u_2, \dots, u_{n-1/2}, u_n$, then the coefficient matrix of the resulting linear system

$$\begin{bmatrix}
a_{00} & a_{01} & & & \\
a_{10} & a_{11} & a_{12} & & \\
& a_{21} & a_{22} & a_{23} & \\
& & \ddots & \ddots & \ddots
\end{bmatrix}$$

is a symmetric tridiagonal matrix, where

$$\begin{aligned}
a_{00} &= \frac{2p_{1/4}}{h_1} + \frac{2p_{3/4}}{h_1} + \frac{h_1}{2} q_{1/2}, & a_{01} &= a_{10} = -\frac{2p_{3/4}}{h_1}, \\
a_{11} &= \frac{2p_{3/4}}{h_1} + \frac{2p_{5/4}}{h_2} + \frac{h_1 + h_2}{4} q_1, & a_{12} &= a_{21} = -\frac{2p_{5/4}}{h_2}, \\
a_{22} &= \frac{2p_{5/4}}{h_2} + \frac{2p_{7/4}}{h_2} + \frac{h_2}{2} q_{3/2}, & a_{23} &= a_{32} = -\frac{2p_{7/4}}{h_2}, \\
a_{33} &= \frac{2p_{7/4}}{h_2} + \frac{2p_{9/4}}{h_3} + \frac{h_2 + h_3}{4} q_2, & a_{34} &= a_{43} = -\frac{2p_{9/4}}{h_3}, \dots
\end{aligned}$$

2.3.3 Convergence order estimates

Inspired by (2.3.2) and (2.3.3), we introduce the following discrete norms

$$|u_h|_{0,h} = \left\{ \sum_{i=1}^n h_i (u_{i-1}^2 + u_{i-1/2}^2 + u_i^2) \right\}^{1/2}, \quad (2.3.9)$$

$$|u_h|_{1,h} = \left\{ \sum_{i=1}^n [(u_{i-1/2} - u_{i-1})^2 + (u_i - u_{i-1/2})^2] / h_i \right\}^{1/2}. \quad (2.3.10)$$

Theorem 2.3.1 *Within U_h , the norms $|\cdot|_{0,h}$ and $|\cdot|_{1,h}$ are equivalent to $|\cdot|_0$ and $|\cdot|_1$ respectively, namely, there exist positive constants C_1, C_2, C_3 and C_4 independent of U_h such that*

$$C_1 |u_h|_{0,h} \leq |u_h|_0 \leq C_2 |u_h|_{0,h}, \quad \forall u_h \in U_h, \quad (2.3.11)$$

$$C_3 |u_h|_{1,h} \leq |u_h|_1 \leq C_4 |u_h|_{1,h}, \quad \forall u_h \in U_h, \quad (2.3.12)$$

Proof By (2.3.3)

$$|u_h|_1^2 = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (u_h')^2 dx = \sum_{i=1}^n h_i \delta_i^T A \delta_i,$$

where

$$\delta_i = \begin{bmatrix} (u_{i-1/2} - u_{i-1})/h_i \\ (u_i - u_{i-1/2})/h_i \end{bmatrix}, \quad A = G^T A_0 G,$$

$$G = \begin{bmatrix} -4 & 4 \\ 3 & -1 \end{bmatrix}, \quad A_0 = \int_0^1 \begin{bmatrix} \xi^2 & \xi \\ \xi & 1 \end{bmatrix} d\xi = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}.$$

A is positive definite since A_0 is positive definite and G is nonsingular. Thus there exist positive constants C_3 and C_4 which are independent of U_h and satisfy

$$C_3^2 \delta_i^T \delta_i \leq \delta_i^T A \delta_i \leq C_4^2 \delta_i^T \delta_i.$$

This implies (2.3.12). Similarly one can prove (2.3.11). This completes the proof. $\square$

Define an interpolation projector $\Pi_h^* : U \rightarrow V_h$ as

$$\Pi_h^* u = \sum_{j=1}^n (u_{j-1/2} \psi_{j-1/2} + u_j \psi_j), \quad \forall u \in U. \quad (2.3.13)$$

Theorem 2.3.2 *For sufficiently small h , $a(u_h, \Pi_h^* u_h)$ is positive definite, that is, there exists a positive constant α such that*

$$a(u_h, \Pi_h^* u_h) \geq \alpha |u_h|_1^2, \quad \forall u_h \in U_h. \quad (2.3.14)$$

Proof First we show that $a_h(u_h, \Pi_h^* u_h)$ is positive definite:

$$\begin{aligned} & a_h(u_h, \Pi_h^* u_h) \\ &= \sum_{j=1}^n [u_{j-1/2} a_h(u_h, \psi_{j-1/2}) + u_j a_h(u_h, \psi_j)] \\ &= \sum_{j=1}^n [2p_{j-3/4} (u_{j-1/2} - u_{j-1})^2 / h_j \\ & \quad + 2p_{j-1/4} (u_j - u_{j-1/2})^2 / h_j] \\ & \quad + \sum_{j=1}^n \left(\frac{h_j}{2} q_{j-1/2} u_{j-1/2}^2 + \frac{h_j + h_{j+1}}{4} q_j u_j^2 \right) \\ &\geq 2p_{\min} |u_h|_{1,h}^2 \geq \alpha_0 |u_h|_1^2 \quad (\alpha_0 > 0). \end{aligned} \quad (2.3.15)$$

Next, we deal with $a(u_h, \Pi_h^* u_h)$. Notice

$$\begin{aligned} & |a(u_h, \Pi_h^* u_h) - a_h(u_h, \Pi_h^* u_h)| \\ &= \left| \sum_{j=1}^n u_{j-1/2} \left(\int_{x_{j-3/4}}^{x_{j-1/4}} q u_h dx - \frac{h_j}{2} q_{j-1/2} u_{j-1/2} \right) \right. \\ & \quad \left. + \sum_{j=1}^n u_j \left(\int_{x_{j-1/4}}^{x_{j+1/4}} q u_h dx - \frac{h_j + h_{j+1}}{4} q_j u_j \right) \right| \\ &\leq \left\{ \sum_{j=1}^n \left[\left(\int_{x_{j-3/4}}^{x_{j-1/4}} (q u_h - q_{j-1/2} u_{j-1/2}) dx \right)^2 \right. \right. \\ & \quad \left. \left. + \left(\int_{x_{j-1/4}}^{x_{j+1/4}} (q u_h - q_j u_j) dx \right)^2 \right] \right\}^{1/2} \left\{ \sum_{j=1}^n (u_{j-1/2}^2 + u_j^2) \right\}^{1/2}. \end{aligned} \quad (2.3.16)$$

By the Cauchy inequality we have

$$\begin{aligned}
|qu_h - q_{j-1/2}u_{j-1/2}|^2 &= \left| \int_{x_{j-1/2}}^x (qu_h)' dx \right|^2 \\
&\leq \frac{h_j}{2} \int_{x_{j-3/4}}^{x_{j-1/4}} [(qu_h)']^2 dx, \quad x_{j-3/4} \leq x \leq x_{j-1/4}, \\
&\quad \left[\int_{x_{j-3/4}}^{x_{j-1/4}} (qu_h - q_{j-1/2}u_{j-1/2}) dx \right]^2 \\
&\leq \frac{h_j}{2} \int_{x_{j-3/4}}^{x_{j-1/4}} (qu_h - q_{j-1/2}u_{j-1/2})^2 dx \\
&\leq \frac{h_j^3}{8} \int_{x_{j-3/4}}^{x_{j-1/4}} [(qu_h)']^2 dx.
\end{aligned}$$

Similarly,

$$\left[\int_{x_{j-1/4}}^{x_{j+1/4}} (qu_h - q_j u_j) dx \right]^2 \leq \frac{h_j^3}{8} \int_{x_{j-1/4}}^{x_{j+1/4}} [(qu_h)']^2 dx.$$

So we have

$$\begin{aligned}
&\left\{ \sum_{j=1}^n \left[\left(\int_{x_{j-3/4}}^{x_{j-1/4}} (qu_h - q_{j-1/2}u_{j-1/2}) dx \right)^2 \right. \right. \\
&\quad \left. \left. + \left(\int_{x_{j-1/4}}^{x_{j+1/4}} (qu_h - q_j u_j) dx \right)^2 \right] \right\}^{1/2} \\
&\leq \left\{ \sum_{j=1}^n \frac{h_j^3}{8} \left(\int_{x_{j-3/4}}^{x_{j-1/4}} [(qu_h)']^2 dx + \int_{x_{j-1/4}}^{x_{j+1/4}} [(qu_h)']^2 dx \right) \right\}^{1/2} \\
&\leq Ch^{3/2} |qu_h|_1.
\end{aligned} \tag{2.3.17}$$

Notice $q \in C^1(I)$ and the equivalence of the seminorm $|\cdot|_1$ and the full norm $\|\cdot\|_1$ in $H_E^1(I)$, then we have

$$|qu_h|_1 \leq |q'u_h|_0 + |qu_h'|_0 \leq C|u_h|_1. \tag{2.3.18}$$

It follows from the quasi-uniformity of the grid and (2.3.11) that

$$\begin{aligned} \left\{ \sum_{j=1}^n (u_{j-1/2}^2 + u_j^2) \right\}^{1/2} &\leq Ch^{-1/2} |u_h|_{0,h} \\ &\leq Ch^{-1/2} |u_h|_0 \leq Ch^{-1/2} |u_h|_1. \end{aligned} \quad (2.3.19)$$

Combining (2.3.16)-(2.3.19) leads to

$$|a(u_h, \Pi_h^* u_h) - a_h(u_h, \Pi_h^* u_h)| \leq Ch |u_h|_1^2.$$

This together with (2.3.15) implies the desired result. $\square$

Theorem 2.3.3 Suppose $u \in H^3(I)$ and u_h are the solutions of the problem (2.3.1) and the quadratic element difference scheme (2.3.4) respectively, then the following estimate holds:

$$|u - u_h|_1 \leq Ch^2 |u|_3. \quad (2.3.20)$$

Proof Noticing

$$a(u, v_h) = (f, v_h), \quad \forall v_h \in V_h,$$

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h,$$

we have

$$a(u - u_h, v_h) = 0, \quad \forall v_h \in V_h. \quad (2.3.21)$$

Let $\Pi_h u$ be the interpolation projection of u onto the trial space U_h . Then using (2.3.14) and (2.3.21) we find that

$$\begin{aligned} |u_h - \Pi_h u|_1^2 &\leq \frac{1}{\alpha} a(u_h - \Pi_h u, \Pi_h^* (u_h - \Pi_h u)) \\ &= \frac{1}{\alpha} a(u - \Pi_h u, \Pi_h^* (u_h - \Pi_h u)). \end{aligned}$$

This gives

$$|u_h - \Pi_h u|_1 \leq \frac{1}{\alpha} \sup_{w_h \in U_h} \frac{|a(u - \Pi_h u, \Pi_h^* w_h)|}{|w_h|_1}. \quad (2.3.22)$$

Notice

$$\begin{aligned}
& a(u - \Pi_h u, \Pi_h^* w_h) \\
&= \sum_{j=1}^n [w_{j-1/2} a(u - \Pi_h u, \psi_{j-1/2}) + w_j a(u - \Pi_h u, \psi_j)] \\
&= \sum_{j=1}^n \left\{ w_{j-1/2} [p_{j-3/4} (u - \Pi_h u)'_{j-3/4} - p_{j-1/4} (u - \Pi_h u)'_{j-1/4}] \right. \\
&\quad \left. + w_{j-1/2} \int_{x_{j-3/4}}^{x_{j-1/4}} q(u - \Pi_h u) dx \right\} \\
&\quad + \sum_{j=1}^n \left\{ w_j [p_{j-1/4} (u - \Pi_h u)'_{j-1/4} - p_{j+1/4} (u - \Pi_h u)'_{j+1/4}] \right. \\
&\quad \left. + w_j \int_{x_{j-1/4}}^{x_{j+1/4}} q(u - \Pi_h u) dx \right\} \\
&= \sum_{j=1}^n [p_{j-3/4} (u - \Pi_h u)'_{j-3/4} (w_{j-1/2} - w_{j-1}) \\
&\quad + p_{j-1/4} (u - \Pi_h u)'_{j-1/4} (w_j - w_{j-1/2})] \\
&\quad + \sum_{j=1}^n \left[w_{j-1/2} \int_{x_{j-3/4}}^{x_{j-1/4}} q(u - \Pi_h u) dx \right. \\
&\quad \left. + w_j \int_{x_{j-1/4}}^{x_{j+1/4}} q(u - \Pi_h u) dx \right].
\end{aligned}$$

Then, we use the Cauchy inequality and (2.3.12), (2.3.19) to get

$$\begin{aligned}
& a(u - \Pi_h u, \Pi_h^* w_h) \\
&\leq |p|_{\max} \left\{ \sum_{j=1}^n [(u - \Pi_h u)'_{j-3/4}]^2 + [(u - \Pi_h u)'_{j-1/4}]^2 \right\}^{1/2} \\
&\quad \cdot \left\{ \sum_{j=1}^n [(w_{j-1/2} - w_{j-1})^2 + (w_j - w_{j-1/2})^2] \right\}^{1/2}
\end{aligned}$$

$$\begin{aligned}
& + |q|_{\max} \left\{ \sum_{j=1}^n \left[\left(\int_{x_{j-3/4}}^{x_{j-1/4}} |u - \Pi_h u| dx \right)^2 \right. \right. \\
& \left. \left. + \left(\int_{x_{j-1/4}}^{x_{j+1/4}} |u - \Pi_h u| dx \right)^2 \right] \right\}^{1/2} \left\{ \sum_{j=1}^n (w_{j-1/2}^2 + w_j^2) \right\}^{1/2} \\
& \leq Ch^{1/2} \left\{ \sum_{j=1}^n [(u - \Pi_h u)'_{j-3/4}]^2 + [(u - \Pi_h u)'_{j-1/4}]^2 \right\}^{1/2} |w_h|_1 \\
& \quad + Ch^{-1/2} \left\{ \sum_{j=1}^n \left[\left(\int_{x_{j-3/4}}^{x_{j-1/4}} |u - \Pi_h u| dx \right)^2 \right. \right. \\
& \quad \left. \left. + \left(\int_{x_{j-1/4}}^{x_{j+1/4}} |u - \Pi_h u| dx \right)^2 \right] \right\}^{1/2} |w_h|_1.
\end{aligned} \tag{2.3.23}$$

The interpolation property gives

$$u - \Pi_h u = 0, \text{ when } x = x_{j-1}, x_{j-1/2}, x_j.$$

So by Rolle theorem there exist $\eta_1 \in (x_{j-1}, x_{j-1/2})$, $\eta_2 \in (x_{j-1/2}, x_j)$ and $\eta_3 \in (\eta_1, \eta_2)$ satisfying

$$(u - \Pi_h u)'(\eta_i) = 0, \quad i = 1, 2,$$

$$(u - \Pi_h u)''(\eta_3) = 0.$$

Thus,

$$\begin{aligned}
(u - \Pi_h u)'_{j-3/4} &= \int_{\eta_1}^{x_{j-3/4}} (u - \Pi_h u)'' dx \\
&= \int_{\eta_1}^{x_{j-3/4}} \left(\int_{\eta_3}^x (u - \Pi_h u)''' dx \right) dx, \\
|(u - \Pi_h u)'_{j-3/4}| &\leq h \int_{x_{j-1}}^{x_j} |u'''| dx \leq h^{3/2} \left(\int_{x_{j-1}}^{x_j} |u'''|^2 dx \right)^{1/2}.
\end{aligned}$$

Similarly we have

$$|(u - \Pi_h u)'_{j-1/4}| \leq h^{3/2} \left(\int_{x_{j-1}}^{x_j} |u'''|^2 dx \right)^{1/2}.$$

Hence

$$\left\{ \sum_{j=1}^n [((u - \Pi_h u)'_{j-3/4})^2 + ((u - \Pi_h u)'_{j-1/4})^2] \right\}^{1/2} \leq Ch^{3/2} |u|_3. \quad (2.3.24)$$

On the other hand,

$$\begin{aligned} & \left\{ \sum_{j=1}^n \left[\left(\int_{x_{j-3/4}}^{x_{j-1/4}} |u - \Pi_h u| dx \right)^2 + \left(\int_{x_{j-1/4}}^{x_{j+1/4}} |u - \Pi_h u| dx \right)^2 \right] \right\}^{1/2} \\ & \leq \left\{ \sum_{j=1}^n h \left[\int_{x_{j-3/4}}^{x_{j-1/4}} |u - \Pi_h u|^2 dx + \int_{x_{j-1/4}}^{x_{j+1/4}} |u - \Pi_h u|^2 dx \right] \right\}^{1/2} \\ & \leq h^{1/2} |u - \Pi_h u|_0 \leq h^{7/2} |u|_3. \end{aligned} \quad (2.3.25)$$

Substituting (2.3.24) and (2.3.25) into (2.3.23) yields

$$a(u - \Pi_h u, \Pi_h^* w_h) \leq Ch^2 |u|_3 |w_h|_1,$$

which together with (2.3.22) implies

$$|u_h - \Pi_h u|_1 \leq Ch^2 |u|_3.$$

This together with the interpolation estimate

$$|u - \Pi_h u|_1 \leq Ch^2 |u|_3$$

gives (2.3.20) and completes the proof. $\square$

2.4 Cubic Element Difference Schemes

Consider a more general two point boundary value problem

$$\begin{cases} Lu \equiv -\frac{d}{dx} \left(p \frac{du}{dx} \right) + r \frac{du}{dx} + qu = f, & a < x < b, \\ u(a) = 0, \quad u'(b) = 0, \end{cases} \quad \begin{matrix} (2.4.1a) \\ (2.4.1b) \end{matrix}$$

where $p \in C^1(I)$, $p \geq p_{\min} > 0$, $q, r \in C(I)$, $f \in L^2(I)$.

In this section we shall deduce a generalized difference scheme with higher accuracy by choosing trial and test spaces as the cubic finite element space of Hermite type and the piecewise linear function space respectively. As in the last two sections, a numerical analysis indicates that this method enjoys the same convergence order as the usual cubic finite element method while its computation is more economical.

2.4.1 Trial and test spaces

As in §2.2 we place a quasi-uniform grid T_h with nodes

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

The trial space U_h is chosen as the cubic finite element space of Hermite type, so each function u_h in U_h satisfies the following condition:

- (i) $u_h \in C^1(I)$, $u_h(a) = 0$,
- (ii) u_h is a cubic polynomial on each element $I_i = [x_{i-1}, x_i]$ and is determined uniquely by its values and derivatives at the two endpoints of I_i .

U_h is a $(2n+1)$ -dimensional subspace of $U = H^2(I) \cap H_E^1(I)$. To construct the basis functions, we first seek for the cubic polynomials to satisfy

$$N_0(0) = 1, N'_0(0) = N_0(1) = N'_0(1) = 0,$$

$$N'_1(0) = 1, N_1(0) = N_1(1) = N'_1(1) = 0.$$

It easily follows that

$$N_0(\xi) = (1 - \xi)^2(2\xi + 1),$$

$$N_1(\xi) = c\xi(1 - \xi)^2. \quad \left(c = \left(\frac{d\xi}{dx}\right)^{-1}\right).$$

Setting $\xi = (x - x_i)/h_{i+1}$ in $N_0(\xi)$ and $N_1(\xi)$, we obtain the right halves of the two basis functions $\phi_i^{(0)}(x)$ and $\phi_i^{(1)}(x)$ respectively.

Similarly, setting $\xi = (x_i - x)/h_i$ we get the left halves. So we have

$$\phi_i^{(0)}(x) = \begin{cases} (1 - h_i^{-1}|x - x_i|)^2(2h_i|x - x_i| + 1), \\ x_{i-1} \leq x \leq x_i, \\ (1 - h_{i+1}^{-1}|x - x_i|)^2(2h_{i+1}|x - x_i| + 1), \\ x_i \leq x \leq x_{i+1}, \\ 0, \quad \text{elsewhere,} \end{cases}$$

$$\phi_i^{(1)}(x) = \begin{cases} (x - x_i)(h_i^{-1}|x - x_i| - 1)^2, \\ x_{i-1} \leq x \leq x_i, \\ (x - x_i)(h_{i+1}^{-1}|x - x_i| - 1)^2, \\ x_i \leq x \leq x_{i+1}, \\ 0, \quad \text{elsewhere.} \end{cases}$$

Any $u_h \in U_h$ can be written uniquely as

$$u_h = \sum_{i=0}^n [u_i \phi_i^{(0)}(x) + u'_i \phi_i^{(1)}(x)],$$

where $u_0 = 0$, $u_i = u_h(x_i)$, $u'_i = u'_h(x_i)$.

On the element $I_i = [x_{i-1}, x_i]$

$$\begin{aligned} u_h &= u_{i-1}(1 - \xi)^2(2\xi + 1) + u_i \xi^2(3 - 2\xi) \\ &\quad + u'_{i-1} h_i \xi(1 - \xi)^2 + u'_i h_i (\xi - 1) \xi^2 \\ &= (\xi^3, \xi^2, \xi, 1) \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{i-1} \\ u_i \\ h_i u'_{i-1} \\ h_i u'_i \end{bmatrix}, \end{aligned} \quad (2.4.2)$$

$$\begin{aligned} u'_h &= u_{i-1} h_i^{-1} (6\xi^2 - 6\xi) + u_i h_i^{-1} (-6\xi^2 + 6\xi) \\ &\quad + u'_{i-1} (3\xi^2 - 4\xi + 1) + u'_i (3\xi^2 - 2\xi) \\ &= (\xi^2, \xi, 1) \begin{bmatrix} -6 & 3 & 3 \\ 6 & -4 & -2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} (u_i - u_{i-1})/h_i \\ u'_{i-1} \\ u'_i \end{bmatrix}, \end{aligned} \quad (2.4.3)$$

where $\xi = (x - x_{i-1})/h_i$.

Now we place a dual grid T_h^* with nodes

$$a = x_0 < x_{1/2} < x_{3/2} < \cdots < x_{n-1/2} < x_n = b,$$

where $x_{j-1/2} = (x_j + x_{j-1})/2$. The test function space V_h is chosen as the piecewise linear function space with basis functions at the point x_j :

$$\psi_j^{(0)}(x) = \begin{cases} 1, & x_{j-1/2} \leq x \leq x_{j+1/2}, \\ 0, & \text{elsewhere,} \end{cases}$$

$$\psi_j^{(1)}(x) = \begin{cases} x - x_j, & x_{j-1/2} \leq x \leq x_{j+1/2}, \\ 0, & \text{elsewhere.} \end{cases}$$

Any $v_h \in V_h$ has a unique expression

$$v_h = \sum_{j=0}^n [v_j \psi_j^{(0)}(x) + v'_j \psi_j^{(1)}(x)],$$

where $v_0 = 0$, $v_i = v_h(x_i)$, $v'_i = v'_h(x_i)$. V_h is clearly a $(2n+1)$ -dimensional subspace of $L^2(I)$. Typical basis functions of U_h and V_h are shown in Fig. 2.4.1.

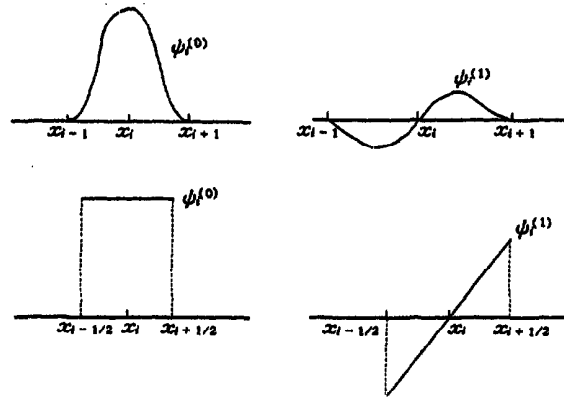


Fig. 2.4.1

2.4.2 Generalized difference methods

The cubic element difference scheme approximating (2.4.1) is: Find $u_h \in U_h$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad (2.4.4)$$

or

$$\begin{cases} a(u_h, \psi_j^{(0)}) = (f, \psi_j^{(0)}), & j = 1, 2, \dots, n, \\ a(u_h, \psi_j^{(1)}) = (f, \psi_j^{(1)}), & j = 0, 1, \dots, n. \end{cases} \quad (2.4.4)'$$

Now we deal with the high order part of $a(u_h, v_h)$

$$b(u_h, v_h) \equiv \int_a^b p \frac{du_h}{dx} \frac{dv_h}{dx} dx.$$

Use (2.4.2), (2.4.3) and note $\xi = 1/2$ when $x = x_{j-1/2}$, then we have

$$u_{j-1/2} = u_h(x_{j-1/2}) = \frac{1}{2}u_{j-1} + \frac{1}{2}u_j + \frac{1}{8}h_j u'_{j-1} - \frac{1}{8}h_j u'_j, \quad (2.4.5)$$

$$j = 1, 2, \dots, n,$$

$$u'_{j-1/2} = u'_h(x_{j-1/2}) = -\frac{3}{2}h_j^{-1}u_{j-1} + \frac{3}{2}h_j^{-1}u_j - \frac{1}{4}u'_{j-1} - \frac{1}{4}u'_j, \quad (2.4.6)$$

$$j = 1, 2, \dots, n.$$

So we obtain for $j = 1, 2, \dots, n-1$,

$$\begin{aligned} b(u_h, \psi_j^{(0)}) &= \int_a^b p \frac{du_h}{dx} \frac{d\psi_j^{(0)}}{dx} dx \\ &= p_{j-1/2} u'_{j-1/2} - p_{j+1/2} u'_{j+1/2} \\ &= \frac{3}{2} p_{j-1/2} (u_j - u_{j-1}) / h_j - \frac{3}{2} p_{j+1/2} (u_{j+1} - u_j) / h_{j+1} \\ &\quad - \frac{1}{4} p_{j-1/2} u'_{j-1} + \frac{1}{4} (p_{j+1/2} - p_{j-1/2}) u'_j + \frac{1}{4} p_{j+1/2} u'_{j+1}, \end{aligned} \quad (2.4.7)$$

$$\begin{aligned} b(u_h, \psi_n^{(0)}) &= \int_a^b p \frac{du_h}{dx} \frac{d\psi_n^{(0)}}{dx} dx = p_{n-1/2} u'_{j-1/2} \\ &= \frac{3}{2} p_{n-1/2} (u_n - u_{n-1}) / h_n - \frac{1}{4} p_{n-1/2} (u'_{n-1} + u'_n), \end{aligned} \quad (2.4.8)$$

$$\begin{aligned}
b(u_h, \psi_j^{(1)}) &= \int_a^b p \frac{du_h}{dx} \frac{d\psi_j^{(1)}}{dx} dx \\
&= -\frac{h_j}{2} p_{j-1/2} u'_{j-1/2} - \frac{h_{j+1}}{2} p_{j+1/2} u'_{j+1/2} + \int_{x_{j-1/2}}^{x_{j+1/2}} p u'_h dx \\
&= -\frac{3}{4} p_{j-1/2} (u_j - u_{j-1}) - \frac{3}{4} p_{j+1/2} (u_{j+1} - u_j) \\
&\quad + \frac{1}{8} p_{j-1/2} h_j u'_{j-1} + \frac{1}{8} (p_{j-1/2} h_j + p_{j+1/2} h_{j+1}) u'_j \\
&\quad + \frac{1}{8} p_{j+1/2} h_{j+1} u'_{j+1} + \int_{x_{j-1/2}}^{x_{j+1/2}} p u'_h dx.
\end{aligned} \tag{2.4.9}$$

Let us approximate the last integral in (2.4.9) in a symmetric fashion:

$$\begin{aligned}
\int_{x_{j-1/2}}^{x_{j+1/2}} p u'_h dx &= p u_h|_{x_{j-1/2}}^{x_{j+1/2}} - \int_{x_{j-1/2}}^{x_{j+1/2}} p' u_h dx \\
&\approx p_{j+1/2} u_{j+1/2} - p_{j-1/2} u_{j-1/2} - u_j (p_{j+1/2} - p_{j-1/2}) \\
&= \frac{1}{2} p_{j-1/2} (u_j - u_{j-1}) + \frac{1}{2} p_{j+1/2} (u_{j+1} - u_j) - \frac{1}{8} p_{j-1/2} h_j u'_{j-1} \\
&\quad + \frac{1}{8} (p_{j-1/2} h_j + p_{j+1/2} h_{j+1}) u'_j - \frac{1}{8} p_{j+1/2} h_{j+1} u'_{j+1}.
\end{aligned}$$

Insert this into (2.4.9) to get an approximation of $b(u_h, \psi_j^{(1)})$:

$$\begin{aligned}
b_h(u_h, \psi_j^{(1)}) &= -\frac{1}{4} p_{j-1/2} (u_j - u_{j-1}) - \frac{1}{4} p_{j+1/2} (u_{j+1} - u_j) \\
&\quad + \frac{1}{4} (p_{j-1/2} h_j + p_{j+1/2} h_{j+1}) u'_j, \quad j = 1, 2, \dots, n-1.
\end{aligned} \tag{2.4.10}$$

Similarly for the two endpoints:

$$\begin{aligned}
b_h(u_h, \psi_0^{(1)}) &= -\frac{h_1}{2} p_{1/2} u'_{1/2} + p_{1/2} u_{1/2} - p_0 u_0 - u_0 (p_{1/2} - p_0) \\
&= \frac{1}{4} p_{1/2} h_1 u'_0 - \frac{1}{4} p_{1/2} u_1,
\end{aligned} \tag{2.4.11}$$

$$\begin{aligned}
b_h(u_h, \psi_n^{(1)}) &= -\frac{h_n}{2} p_{n-1/2} u'_{n-1/2} \\
&\quad + p_n u_n - p_{n-1/2} u_{n-1/2} - u_n (p_n - p_{n-1/2}) \\
&= -\frac{1}{4} p_{n-1/2} (u_n - u_{n-1}) + \frac{1}{4} p_{n-1/2} h_n u'_n.
\end{aligned} \tag{2.4.12}$$

If we define $p_{-1/2}, p_{n+1/2}, h_{-1}, h_{n+1}, u_{-1}, u_{n+1}$ and u'_{n+1} as zero, then (2.4.8), (2.4.11) and (2.4.12) can be expressed in terms of (2.4.7) and (2.4.10) respectively, that is, (2.4.7) is valid for $j = 1, 2, \dots, n$, and (2.4.10) for $j = 0, 1, \dots, n$. Write

$$b_h(u_h, \psi_j^{(0)}) \equiv b(u_h, \psi_j^{(0)}), \quad j = 1, 2, \dots, n,$$

then we have the following bilinear form approximating $b(u_h, v_h)$

$$b_h(u_h, v_h) = (v'_0, v_1, v'_1, \dots, v_n, v'_n) A (u'_0, u_1, u'_1, \dots, u_n, u'_n)^T, \tag{2.4.13}$$

where A is a symmetric matrix of the form

$$\begin{bmatrix}
a_{00} & a_{01} & 0 & & & & & \\
a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & & & \\
0 & a_{21} & a_{22} & a_{23} & 0 & & & \\
& a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & \\
& a_{41} & 0 & a_{43} & a_{44} & a_{45} & 0 & \\
& & & \dots & \dots & \dots & \dots & \dots
\end{bmatrix},$$

where

$$\begin{aligned}
a_{00} &= \frac{1}{4} p_{1/2} h_1, & a_{01} &= a_{10} = -\frac{1}{4} p_{1/2}, \\
a_{11} &= \frac{3}{2} \left(\frac{p_{1/2}}{h_1} + \frac{p_{3/2}}{h_2} \right), & a_{12} &= a_{21} = \frac{1}{4} (p_{3/2} - p_{1/2}), \\
a_{13} &= a_{31} = -\frac{3}{2} \frac{p_{3/2}}{h_2}, & a_{14} &= a_{41} = -a_{23} = -a_{32} = \frac{1}{4} p_{3/2}, \\
a_{22} &= \frac{1}{4} (p_{1/2} h_1 + p_{3/2} h_2), & a_{33} &= \frac{3}{2} \left(\frac{p_{3/2}}{h_2} + \frac{p_{5/2}}{h_3} \right), \\
a_{34} &= a_{43} = \frac{1}{4} (p_{5/2} - p_{3/2}), & a_{35} &= -\frac{3}{2} \frac{p_{5/2}}{h_3}, \\
a_{36} &= -a_{45} = \frac{1}{4} p_{5/2}, & a_{44} &= \frac{1}{4} (p_{3/2} h_2 + p_{5/2} h_3).
\end{aligned}$$

These give the leading terms (corresponding to the highest order derivative) of the matrix of the generalized difference equation. We see that the coefficient matrix of the generalized difference equation is sparse, and of the same bandwidth as the finite element method, but it is much more economical to obtain the discrete equation for the generalized difference method than the finite element method.

2.4.3 Some lemmas

First let us introduce a discrete norm for $u_h \in U_h$:

$$|u_h|_{1,h} = \left\{ \sum_{j=1}^n h_j \left[\left(\frac{u_j - u_{j-1}}{h_j} \right)^2 + u_{j-1}'^2 + u_j'^2 \right] \right\}^{1/2}. \quad (2.4.14)$$

The proof of the following lemma is similar to that of Theorem 2.3.1 by noting (2.4.3).

Lemma 2.4.1 *The norms $|\cdot|_{1,h}$ and $|\cdot|_1$ are equivalent, i.e. there exist constants C_1 and C_2 independent of the subspace U_h such that*

$$C_1 |u_h|_{1,h} \leq |u_h|_1 \leq C_2 |u_h|_{1,h}, \quad \forall u_h \in U_h. \quad (2.4.15)$$

Next we show the positive definiteness of the leading term of the difference equation.

Lemma 2.4.2 *The bilinear form $b(u_h, \Pi_h^* u_h)$ is positive definite, i.e., there exists a constant $\beta > 0$ independent of the subspace U_h such that*

$$b(u_h, \Pi_h^* u_h) \geq \beta |u_h|_1^2, \quad \forall u_h \in U_h. \quad (2.4.16)$$

Proof First we show the positive definiteness of $b_h(u_h, \Pi_h^* u_h)$. By (2.4.7), (2.4.10) and the definition for the quantities outside the end-

points, we have

$$\begin{aligned}
& b_h(u_h, \Pi_h^* u_h) \\
&= \sum_{j=0}^n [u_j b_h(u_h, \psi_j^{(0)}) + u'_j b_h(u_h, \psi_j^{(1)})] \\
&= \sum_{j=1}^n \left[\frac{3}{2} p_{j-1/2} \frac{u_j - u_{j-1}}{h_j} u_j - \frac{3}{2} p_{j+1/2} \frac{u_{j+1} - u_j}{h_{j+1}} u_j \right. \\
&\quad \left. - \frac{1}{4} p_{j-1/2} (u'_{j-1} + u'_j) u_j + \frac{1}{4} p_{j+1/2} (u'_{j+1} + u'_j) u_j \right] \\
&\quad + \sum_{j=0}^n \left[-\frac{1}{4} p_{j-1/2} (u_j - u_{j-1}) u'_j - \frac{1}{4} p_{j+1/2} (u_{j+1} - u_j) u'_j \right. \\
&\quad \left. + \frac{1}{4} p_{j-1/2} h_j u_j'^2 + \frac{1}{4} p_{j+1/2} h_{j+1} u_j'^2 \right] \\
&= \sum_{j=1}^n \left[\frac{3}{2} p_{j-1/2} \frac{u_j - u_{j-1}}{h_j} u_j - \frac{3}{2} p_{j-1/2} \frac{u_j - u_{j-1}}{h_j} u_{j-1} \right. \\
&\quad \left. - \frac{1}{4} p_{j-1/2} (u'_{j-1} + u'_j) u_j + \frac{1}{4} p_{j-1/2} (u'_{j-1} + u'_j) u_{j-1} \right] \\
&\quad + \sum_{j=1}^n \left[-\frac{1}{4} p_{j-1/2} (u_j - u_{j-1}) u'_j - \frac{1}{4} p_{j-1/2} (u_j - u_{j-1}) u'_{j-1} \right. \\
&\quad \left. + \frac{1}{4} p_{j-1/2} h_j u_j'^2 + \frac{1}{4} p_{j-1/2} h_j u_{j-1}'^2 \right] \\
&= \sum_{j=1}^n h_j p_{j-1/2} \left[\frac{3}{2} \left(\frac{u_j - u_{j-1}}{h_j} \right)^2 - \frac{1}{2} \frac{u_j - u_{j-1}}{h_j} (u'_j + u'_{j-1}) \right. \\
&\quad \left. + \frac{1}{4} (u_j'^2 + u_{j-1}'^2) \right] \\
&\geq \frac{1}{8} P_{\min} \sum_{j=1}^n h_j \left[\left(\frac{u_j - u_{j-1}}{h_j} \right)^2 + u_j'^2 + u_{j-1}'^2 \right] \\
&\geq \beta' |u_h|_1^2 \quad (\beta' > 0).
\end{aligned} \tag{2.4.17}$$

Next we estimate

$$\begin{aligned}
 E(u_h, \Pi_h^* u_h) &\equiv b(u_h, \Pi_h^* u_h) - b_h(u_h, \Pi_h^* u_h) \\
 &= \sum_{j=0}^n u'_j \int_{x_{j-1/2}}^{x_{j+1/2}} p'(u_j - u_h) dx \\
 &= \sum_{j=0}^n u'_j \int_{x_{j-1/2}}^{x_{j+1/2}} p' u'_h(\xi_j)(x - x_j) dx.
 \end{aligned}$$

It follows from (2.4.3) that on the interval $I_i = [x_{j-1}, x_j]$

$$|u'_h(x)| \leq C \left(\left| \frac{u_j - u_{j-1}}{h_j} \right| + |u'_{j-1}| + |u'_j| \right).$$

This together with the quasi-uniformity of the grid leads to

$$\begin{aligned}
 \left\{ \sum_{j=0}^n [u'_h(\xi_j)]^2 \right\}^{1/2} &\leq Ch^{-1/2} |u_h|_{1,h}, \\
 \left\{ \sum_{j=0}^n (u'_j)^2 \right\}^{1/2} &\leq Ch^{-1/2} |u_h|_{1,h}.
 \end{aligned}$$

So we have

$$|E(u_h, \Pi_h^* u_h)| \leq Ch |u_h|_1^2. \quad (2.4.18)$$

Combining (2.4.17) and (2.4.18) yields (2.4.16). This completes the proof. $\square$

Denote by P_h the orthogonal projector from L^2 to V_h . Then the cubic element difference scheme (2.4.4) is equivalent to

$$(Lu_h, P_h v) = (f, P_h v), \quad \forall v \in L^2(I), \quad (2.4.19)$$

or

$$L_h u_h = f_h,$$

where $L_h = P_h L$, $f_h = P_h f$.

Lemma 2.4.3 *Suppose the homogeneous equation*

$$a(u, v) = 0, \quad \forall v \in H_E^1(I) \quad (2.4.20)$$

admits only the trivial solution, then there exists a constant $\alpha > 0$ independent of the subspace U_h such that for sufficiently small h

$$|||L_h u_h||| \equiv \sup_{\substack{w_h \in U_h \\ |w_h|_1=1}} |(Lu_h, \Pi_h^* w_h)| \geq \alpha |u_h|_1, \quad \forall u_h \in U_h. \quad (2.4.21)$$

Proof First write L as

$$L = L_1 + L_2, \quad (2.4.22)$$

where

$$\begin{aligned} L_1 u &= -\frac{d}{dx} \left(p \frac{du}{dx} \right) + r \frac{du}{dx} + (q + \lambda)u, \\ L_2 u &= -\lambda u, \end{aligned}$$

where λ is a positive constant to be chosen. We find

$$\begin{aligned} & (L_1 u_h, \Pi_h^* u_h) \\ &= \left(-\frac{d}{dx} \left(p \frac{du_h}{dx} \right), \Pi_h^* u_h \right) + \left(r \frac{du_h}{dx}, \Pi_h^* u_h \right) + ((q + \lambda)u_h, \Pi_h^* u_h) \\ &\geq \beta |u_h|_1^2 - \max_{x \in I} |r| \cdot |u_h|_1 \cdot |\Pi_h^* u_h|_0 + (\lambda - \max_{x \in I} |q|) |u_h|_0^2 \\ &\quad - (\lambda + \max_{x \in I} |q|) |u_h|_0 |\Pi_h^* u_h - u_h|_0. \end{aligned} \quad (2.4.23)$$

By the interpolation theory

$$|\Pi_h^* u_h - u|_0 \leq Ch^2 |u|_2, \quad \forall u \in U. \quad (2.4.24)$$

The inverse property of the finite elements implies

$$|u_h|_2 \leq Ch^{-1} |u_h|_1 \leq Ch^{-2} |u_h|_0, \quad \forall u_h \in U_h. \quad (2.4.25)$$

Hence

$$|\Pi_h^* u_h|_0 \leq |u_h|_0 + |\Pi_h^* u_h - u_h|_0 \leq C |u_h|_0. \quad (2.4.26)$$

Then we combine (2.4.23)-(2.4.26) to obtain

$$(L_1 u_h, \Pi_h^* u_h) \geq \beta |u_h|_1^2 + C_3 |u_h|_0^2 - (C_4 + C_5 h) |u_h|_1 |u_h|_0,$$

where the constants C_3 and C_5 increase with λ . Exploiting the ϵ -inequality $ab \leq \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2$ ($\epsilon > 0$), we see that there exists a positive constant γ such that for sufficiently small h

$$(L_1 u_h, \Pi_h^* u_h) \geq \gamma |u_h|_1^2, \quad \forall u_h \in U_h, \quad (2.4.27)$$

where γ is independent of U_h .

Next we turn to show (2.4.21). Suppose by contradiction that there exists a sequence $\{\tilde{u}_h\}$, $\tilde{u}_h \in U_h$, satisfying

$$|\tilde{u}_h|_1 = 1, \quad |||L_h \tilde{u}_h||| \rightarrow 0 \text{ as } h \rightarrow 0. \quad (2.4.28)$$

Since $H_E^1(I)$ is weakly sequentially compact, $\{\tilde{u}_h\}$ has a subsequence (again written as $\{\tilde{u}_h\}$) which converges weakly to some $\tilde{u} \in H_E^1(I)$. Take any $w \in U$ and write $\Pi_h w$ as the interpolation projection of w onto U_h . It is clear that $\Pi_h^*(w - \Pi_h w) = 0$. It follows from the interpolation theory that when h is sufficiently small

$$|\Pi_h w|_1 \leq |w|_1 + |\Pi_h w - w|_1 \leq |w|_1 + Ch|w|_2 \leq \|w\|_2. \quad (2.4.29)$$

Now by (2.4.28)

$$\begin{aligned} |(L\tilde{u}_h, \Pi_h^* w)| &= |(L\tilde{u}_h, \Pi_h^* \Pi_h w)| \\ &\leq C |||L_h \tilde{u}_h||| \cdot |\Pi_h w|_1 \leq C |||L_h \tilde{u}_h||| \cdot \|w\|_2 \rightarrow 0 \quad (h \rightarrow 0). \end{aligned} \quad (2.4.30)$$

On the other hand, it follows from (2.4.24) and (2.4.25) that

$$\begin{aligned} |(L\tilde{u}_h, \Pi_h^* w - w)| &\leq C \|\tilde{u}_h\|_2 |\Pi_h^* w - w|_0 \\ &\leq Ch \|\tilde{u}_h\|_1 \|w\|_2 \leq Ch \|w\|_2 \rightarrow 0 \quad (h \rightarrow 0). \end{aligned} \quad (2.4.31)$$

Combining (2.4.30) and (2.4.31) leads to

$$a(\tilde{u}_h, w) = (L\tilde{u}_h, w) \rightarrow 0 \quad (h \rightarrow 0). \quad (2.4.32)$$

For fixed $w \in H_E^1(I)$, $a(u, w)$ is a bounded linear functional on $H_E^1(I)$, which implies

$$a(\tilde{u}_h, w) \rightarrow a(\tilde{u}, w) \quad (h \rightarrow 0). \quad (2.4.33)$$

By (2.4.32) and (2.4.33) we have

$$a(\tilde{u}, w) = 0, \quad \forall w \in U. \quad (2.4.34)$$

In fact, the above equality is valid for any $w \in H_E^1(I)$ since U is dense in $H_E^1(I)$. The assumption of the lemma then implies $\tilde{u} = 0$. So the sequence $\{\tilde{u}_h\}$ converges weakly to zero. By the compactness of the imbedding of $H_E^1(I)$ in L^2 we know that $\{\tilde{u}_h\}$ converges strongly to zero in L^2 , which gives

$$|L_2 \tilde{u}_h|_0 \rightarrow 0 \quad (h \rightarrow 0).$$

Furthermore, it follows from (2.4.26) that

$$|(L_2 \tilde{u}_h, \Pi_h^* \tilde{u}_h)| \leq C |L_2 \tilde{u}_h|_0 |\tilde{u}_h|_0 \rightarrow 0 \quad (h \rightarrow 0). \quad (2.4.35)$$

Finally by (2.4.28) and (2.4.35) we conclude

$$\begin{aligned} & |(L_1 \tilde{u}_h, \Pi_h^* \tilde{u}_h)| \\ & \leq |(L \tilde{u}_h, \Pi_h^* \tilde{u}_h)| + |(L_2 \tilde{u}_h, \Pi_h^* \tilde{u}_h)| \\ & \leq |||L_h \tilde{u}_h||| + |(L_2 \tilde{u}_h, \Pi_h^* \tilde{u}_h)| \rightarrow 0 \quad (h \rightarrow 0). \end{aligned} \quad (2.4.36)$$

This contradicts (2.4.27) and completes the proof. $\square$

2.4.4 Existence, uniqueness and stability

Theorem 2.4.1 *Assume that the homogeneous equation (2.4.20) admits only the trivial solution. Then for sufficiently small h , the cubic element difference scheme (2.4.4) has a unique solution for any given $f \in L^2(I)$.*

Proof By virtue of the well-known results in linear algebra theories, one only needs to show that the homogeneous equation

$$L_h u_h = 0$$

admits solely the trivial solution, which follows from (2.4.21). $\square$

The stability problem of the scheme considers the difference between the solutions of equation (2.4.19) and its perturbed equation

$$(L_h + E_h)\hat{u}_h = f_h + g_h, \quad (2.4.37)$$

where $E_h = P_h E$, E being a linear perturbation of L , and $g_h = P_h g$, g a perturbation of f .

Definition 2.4.1 Let equation (2.4.19) be uniquely solvable, for any $f \in L^2(I)$ and $0 < h \leq h_0$. (2.4.19) is said to be stable, if there exist positive constants α_0 , β_0 and δ_0 independent of the subspace U_h and the function f such that the perturbed equation (2.4.37) always has a unique solution $\hat{u}_h \in U_h$ for any $g_h = P_h g \in V_h$, $E_h = P_h E : U_h \rightarrow V_h$, and $|||E_h||| \leq \delta_0$, provided $0 < h \leq h_0$; and that this solution satisfies

$$|\hat{u}_h - u_h|_1 \leq \alpha_0 |||E_h||| |u_h|_1 + \beta_0 |||g_h|||, \quad (2.4.38)$$

where

$$\begin{aligned} |||E_h||| &= \sup_{\substack{u_h \in U_h \\ |u_h|_1=1}} |||E_h u_h|||, \\ |||E_h u_h||| &= \sup_{\substack{w_h \in U_h \\ |w_h|_1=1}} |(E u_h, \Pi_h^* w_h)|, \\ |||g_h||| &= \sup_{\substack{w_h \in U_h \\ |w_h|_1=1}} |(g, \Pi_h^* w_h)|. \end{aligned}$$

Theorem 2.4.2 Let the conditions of Theorem 2.4.1 hold. Then the cubic element difference scheme (2.4.4) is stable.

Proof By Theorem 2.4.1 we know that (2.4.19), and hence (2.4.4), is uniquely solvable. Choose δ_0 such that $0 < \delta_0 < \alpha$. Then it follows from Lemma 2.4.3 that there exists h_0 such that for $0 < h \leq h_0$ and $|||E_h||| < \delta_0$ we have

$$\begin{aligned} |||(L_h + E_h)u_h||| &= \sup_{\substack{w_h \in U_h \\ |w_h|_1=1}} |((L_h + E_h)u_h, \Pi_h^* w_h)| \\ &\geq \alpha |u_h|_1 - |||E_h u_h||| \geq (\alpha - \delta_0) |u_h|_1, \quad \forall u_h \in U_h. \end{aligned} \quad (2.4.39)$$

Thus (2.4.37) has a unique solution $\hat{u}_h$. By (2.4.19) and (2.4.37) we have

$$(L_h + E_h)(\hat{u}_h - u_h) = g_h - E_h u_h.$$

Hence,

$$\begin{aligned} (\alpha - \delta_0)|\hat{u}_h - u_h|_1 &\leq |||(L_h + E_h)(\hat{u}_h - u_h)||| \\ &= |||g_h - E_h u_h||| \leq |||g_h||| + |||E_h||| \cdot |u_h|_1. \end{aligned}$$

Therefore, (2.4.38) holds for $\alpha_0 = \beta_0 = (\alpha - \delta_0)^{-1}$. $\square$

2.4.5 Convergence order estimates

Theorem 2.4.3 *Let the conditions of Theorem 2.4.1 be satisfied, and let u be the solution of (2.4.1) satisfying $u \in H^4(I)$ and u_h the solution of the cubic element difference scheme (2.4.4). Then the following error estimate holds for sufficiently small h :*

$$|u - u_h|_1 \leq Ch^3|u|_4. \quad (2.4.40)$$

Proof Clearly we have

$$(Lu, \Pi_h^* w_h) = (Lu_h, \Pi_h^* w_h), \quad \forall w_h \in U_h. \quad (2.4.41)$$

By Lemma 2.4.3, (2.4.41), (2.4.24) and (2.4.25) we deduce that

$$\begin{aligned} &|\Pi_h u - u_h|_1 \\ &\leq C |||L_h(\Pi_h u - u)||| \\ &\leq C \sup_{\substack{w_h \in U_h \\ |w_h|_1=1}} |(L(\Pi_h u - u), \Pi_h^* w_h)| \\ &\leq C \sup_{\substack{w_h \in U_h \\ |w_h|_1=1}} \{|(L(\Pi_h u - u), w_h)| + |(L(\Pi_h u - u), w_h - \Pi_h^* w_h)|\} \\ &\leq C(|\Pi_h u - u|_1 + \sup_{\substack{w_h \in U_h \\ |w_h|_1=1}} h^2 \|\Pi_h u - u\|_2 |w_h|_2) \\ &\leq C(|\Pi_h u - u|_1 + h \|\Pi_h u - u\|_2) \\ &\leq Ch^3|u|_4. \end{aligned} \quad (2.4.42)$$

This leads to (2.4.40) and completes the proof. $\square$

We remark that the error estimate (2.4.40) is of the optimal order precisely as the cubic finite element method of Hermite type.

Definition 2.4.2 *In the deduction of the error estimates of generalized difference methods in §2.2, §2.3 and this section, the key point is to show that the bilinear forms $a(u_h, \Pi_h^* u_h)$ ($u_h \in U_h$) satisfy the inequalities (2.2.9), (2.3.14) or (2.4.16) respectively. Henceforth in such a case we say that the bilinear form $a(u_h, \Pi_h^* u_h)$ is uniformly positive definite or U_h -elliptic.*

2.4.6 Numerical examples

The usual second order central difference method (FD), the cubic finite element method (FE) and the above three generalized difference methods (GD1, GD2, GD3) are used to solve the following boundary value problem

$$\begin{cases} -u''(x) = x^2, & x \in (0, 1), \\ u(0) = 0, & u'(\pi) = 0. \end{cases} \quad \begin{matrix} (2.4.43a) \\ (2.4.43b) \end{matrix}$$

The true solution (TS) of (2.4.43) is

$$u = \frac{1}{3}\pi^3 x - \frac{1}{12}x^4.$$

Take the step length $h = \pi/n$ and write $x_i = i\pi/16$, $i = 1, 2, \dots, 16$. The numerical results of the five methods are given in Table 2.4.1. We recall that (GD) is more economical than (FE). Table 2.4.1 shows that these (GD)'s are much more accurate than (FD), and (GD3) is nearly accurate as (FE).

Table 2.4.1 Numerical results

n	meth.	x_2	x_4	x_6	x_8	x_{10}	x_{12}	x_{14}	x_{16}
2	FD				18.26438				30.44063
	GD1				15.98118				25.36695
	GD2		8.10157		15.79093		21.92656		24.60594
	GD3				15.70229				24.35250
	FE				15.71638				24.35250
4	FD		8.37117		16.36184		22.83047		25.87453
	GD1		8.10157		15.79093		21.92656		24.60594
	GD2	4.05772	8.08968	12.02453	15.74336	19.07971	21.81954	23.70125	24.41569
	GD3		8.08401		15.72572		21.78229		24.35250
	FE		8.08492		15.72666		21.78320		24.35250
8	FD	4.09046	8.15714	12.12869	15.88020	19.26321	22.04567	23.97199	24.73301
	GD1	4.05772	8.08968	12.02453	15.74336	19.07971	21.81954	23.70125	24.41569
	GD2	4.05698	8.08671	12.01784	15.73147	19.06113	21.79279	23.66484	24.36813
	GD3	4.05666	8.08507	12.01560	15.72753	19.05499	21.78395	23.65281	24.35250
	FE	4.05671	8.08513	12.01565	15.72759	19.05505	21.78401	23.65287	24.35250
16	FD	4.06519	8.10363	12.04336	15.76729	19.10714	21.84047	23.73269	24.44763
	GD1	4.05698	8.08671	12.01784	15.73147	19.06113	21.79279	23.66484	24.36813
	GD2	4.05697	8.08596	12.01617	15.72850	19.05649	21.78610	23.65573	24.35624
	GD3	4.05676	8.08578	12.01572	15.72765	19.05511	21.78407	23.65291	24.35250
	FE	4.05676	8.08579	12.01572	15.72765	19.05511	21.78407	23.65292	24.35250
	TS	4.05677	8.08579	12.01572	15.72766	19.05512	21.78407	23.65292	24.35250

2.5 Estimates in L^2 and Maximum Norms

2.5.1 L^2 -estimates

First let us consider the linear element difference scheme introduced in §2.2 for the two point boundary value problem (2.2.1).

Theorem 2.5.1 *Let u_h be the solution to (2.2.6), and u to (2.2.1) with $u \in H_E^1(I) \cap W^{3,1}(I)$. Then the following estimate holds:*

$$\|u - u_h\|_0 \leq Ch^2 \|u\|_{3,1}. \quad (2.5.1)$$

Proof Let us introduce an auxiliary problem: For given $g = u - u_h$, find $w \in H_E^1(I)$ such that

$$a(v, w) = (g, v), \quad \forall v \in H_E^1(I). \quad (2.5.2)$$

By the differential equation theory we know that problem (2.5.2) possesses a unique solution satisfying

$$\|w\|_2 \leq C\|g\|_0. \quad (2.5.3)$$

It follows from (2.5.2) and (2.2.11) that

$$\begin{aligned} \|u - u_h\|_0^2 &= a(u - u_h, w) \\ &= a(u - u_h, w - \Pi_h w) + a(u - u_h, \Pi_h w) - a(u - u_h, \Pi_h^* w). \end{aligned} \quad (2.5.4)$$

By (2.2.10) and (2.5.3) we have

$$\begin{aligned} |a(u - u_h, w - \Pi_h w)| &\leq C|u - u_h|_1|w - \Pi_h w|_1 \\ &\leq Ch^2|u|_2\|u - u_h\|_0. \end{aligned} \quad (2.5.5)$$

Next we compute

$$\begin{aligned} &a(u - u_h, \Pi_h w) \\ &= \int_a^b p(u - u_h)'(\Pi_h w)' dx \\ &= \sum_{j=1}^n \left[\int_{x_{j-1}}^{x_j} (p - p_{j-1/2})(u - u_h)' dx \right. \\ &\quad \left. + \int_{x_{j-1}}^{x_j} p_{j-1/2}(u - u_h)' dx \right] \frac{w_j - w_{j-1}}{h_j}, \end{aligned} \quad (2.5.6)$$

$$\begin{aligned} &a(u - u_h, \Pi_h^* w) \\ &= \sum_{j=1}^n w_j a(u - u_h, \psi_j) \\ &= \sum_{j=1}^n p_{j-1/2}(u - u_h)'_{j-1/2}(w_j - w_{j-1}). \end{aligned} \quad (2.5.7)$$

Thus

$$\begin{aligned} &a(u - u_h, \Pi_h w) - a(u - u_h, \Pi_h^* w) \\ &= \sum_{j=1}^n \int_{x_{j-1}}^{x_j} (p - p_{j-1/2})(u - u_h)' dx \frac{w_j - w_{j-1}}{h_j} \\ &\quad + \sum_{j=1}^n p_{j-1/2}(u_j - u_{j-1} - h_j u'_{j-1/2}) \frac{w_j - w_{j-1}}{h_j}. \end{aligned} \quad (2.5.8)$$

It follows from (2.2.10) and (2.5.3) that

$$\begin{aligned} & \left| \sum_{j=1}^n \int_{x_{j-1}}^{x_j} (p - p_{j-1/2})(u - u_h)' dx \frac{w_j - w_{j-1}}{h_j} \right| \\ & \leq Ch|u - u_h|_1|w|_1 \leq Ch^2|u|_2\|u - u_h\|_0. \end{aligned} \quad (2.5.9)$$

Using the Taylor expansion with an integral remainder we have

$$\begin{aligned} & \left| \sum_{j=1}^n p_{j-1/2}(u_j - u_{j-1} - h_j u'_{j-1/2}) \frac{w_j - w_{j-1}}{h_j} \right| \\ & \leq Ch^2|u|_{3,1}|w|_{1,\infty} \leq Ch^2|u|_{3,1}\|u - u_h\|_0. \end{aligned} \quad (2.5.10)$$

Combining (2.5.4), (2.5.5), (2.5.8)-(2.5.10) and noting the imbedding relation $W^{3,1}(I) \rightarrow H^2(I)$, we have

$$\|u - u_h\|_0^2 \leq Ch^2\|u\|_{3,1}\|u - u_h\|_0.$$

This validates the estimate (2.5.1) and completes the proof. $\square$

Theorem 2.5.1 indicates that the solution of the linear element difference scheme possesses an optimal order estimate in L^2 norm, but it requires higher smoothness of the true solution u than the corresponding finite element method. Maybe it is a reasonable punishment for taking only piecewise constant functions as the test space. The deduction of the L^2 -estimate for the quadratic element difference scheme is rather tedious and is left as an exercise for interested readers.

In the next theorem we give an L^2 estimate for the cubic element difference scheme, which is more like the counterpart for the finite element method since here the test space has a better approximation property.

Theorem 2.5.2 *Let u_h be the solution to the cubic element difference scheme (2.4.4) and u to (2.4.1) with $u \in H_E^1(I) \cap H^4(I)$. Then the following estimate holds:*

$$\|u - u_h\|_0 \leq Ch^4|u|_4. \quad (2.5.11)$$

Proof As in the proof of Theorem 2.5.1 we introduce an auxiliary problem: For given $g = u - u_h$, find $w \in H_E^1(I)$ such that

$$a(v, w) = (g, v), \quad \forall v \in H_E^1(I).$$

This together with (2.4.1) and (2.4.4) implies

$$\begin{aligned} \|u - u_h\|_0^2 &= a(u - u_h, w) \\ &= (L(u - u_h), w - \Pi_h^* w) \\ &\leq C \|u - u_h\|_2 \|w - \Pi_h^* w\|_0. \end{aligned} \quad (2.5.12)$$

By the approximation theory, the inverse property of the finite elements and (2.4.42) we have

$$\begin{aligned} \|u - u_h\|_2 &\leq \|u - \Pi_h u\|_2 + \|\Pi_h u - u_h\|_2 \\ &\leq Ch^2 |u|_4 + Ch^{-1} |\Pi_h - u_h|_1 \\ &\leq Ch^2 |u|_4. \end{aligned} \quad (2.5.13)$$

Also note

$$\|w - \Pi_h^* w\|_0 \leq Ch^2 |w|_2 \leq Ch^2 \|u - u_h\|_0. \quad (2.5.14)$$

(2.5.11) now follows from (2.5.12)-(2.5.14). This completes the proof. $\square$

2.5.2 Maximum norm estimates

The L^2 -estimates easily results in the L^∞ -estimates which indicate the uniform convergence of the approximate solutions to the true solutions.

Theorem 2.5.3 *Under the assumptions of Theorem 2.5.1, the solution of the linear element scheme (2.2.6) satisfies the following error estimate:*

$$\|u - u_h\|_{0,\infty} \leq Ch^{3/2} \|u\|_{3,1}. \quad (2.5.15)$$

Proof Clearly we have

$$\|u - u_h\|_{0,\infty} \leq \|u - \Pi_h u\|_{0,\infty} + \|\Pi_h u - u_h\|_{0,\infty}. \quad (5.16)$$

For the first term in the right side we have, by the Sobolev interpolation theorem, for some $I_i \in T_h$ that

$$\|u - \Pi_h u\|_{0,\infty} = \|u - \Pi_h u\|_{0,\infty,I_i} \leq Ch^{3/2}|u|_{2,I_i} \leq Ch^{3/2}|u|_2. \quad (5.17)$$

For the second term, we use the inverse property of the finite element method and the L^2 -estimate (2.5.1) to obtain

$$\begin{aligned} & \|\Pi_h u - u_h\|_{0,\infty} \\ & \leq Ch^{-1/2}\|\Pi_h u - u_h\|_0 \\ & \leq Ch^{-1/2}(\|u - u_h\|_0 + \|u - \Pi_h u\|_0) \\ & \leq Ch^{3/2}\|u\|_{3,1}. \end{aligned} \quad (2.5.18)$$

Combining (2.5.16)-(2.5.18) yields (2.5.15) and this completes the proof. $\square$

The next theorem can be similarly proved.

Theorem 2.5.4 *Under the assumption of Theorem 2.5.2, the following estimate holds for the cubic element difference scheme (2.4.4)*

$$\|u - u_h\|_{0,\infty} \leq Ch^{7/2}\|u\|_4. \quad (2.5.19)$$

2.6 Superconvergence

In this section we first give an outline of the concept of optimal stress points and then, in particular, we show some superconvergence results for the generalized difference methods for two point boundary value problems.

2.6.1 Optimal stress points

In the error analysis, the determination of the upper bound of $\|u - u_h\|_m$ is usually reduced to the estimation of $\|u - \Pi_h u\|_m$ and $a(u - \Pi_h u, \Pi_h^* w_h)$. By the approximation theory, in general we can only obtain, limited by the degree k of the approximate polynomials, that

$$\|u - \Pi_h u\|_m \leq Ch^{k+1-m} \|u\|_{k+1}.$$

In general this estimate can not be improved even if the solution u possesses an higher smoothness. Therefore,

$$\|u - u_h\|_m = O(h^{k+1-m})$$

is the optimal order error estimate. But this fact does not exclude the possibility that the approximation of the derivatives may be of higher order accuracy at some special points, called optimal stress points. The following definition describes an example of such points.

Definition 2.6.1 *Point x_0 is called a optimal stress point if there exists a $q \in [1, \infty]$ such that*

$$|\bar{\nabla}(u - \Pi_h u)(x_0)| \leq Ch^{k+1-\frac{N}{q}} \|u\|_{k+2,q,E}, \quad \forall u \in W^{k+2,q}(E), \quad (2.6.1)$$

where E denotes the union of all the elements containing x_0 , $\bar{\nabla}v(x_0)$ the arithmetic mean of the values $\nabla v(x_0)$ at every element in E , N the dimension of the region, and C a constant independent of the grid T_h and the solution u .

The superconvergence theory of finite elements has clarified the distribution of the interpolation optimal stress points for some most in use finite elements. For instance, the set of the interpolation optimal stress points for the one-dimensional $\mathcal{P}_k$ type Lagrange element is

$$N_k = F\hat{N}_k,$$

where F is the invertible affine mapping from the reference element $\hat{K} = [-1, 1]$ to the finite element K , and $\hat{N}_k$ is the set of the interpolation optimal stress points on $[-1, 1]$:

$$\hat{N}_1 = \{0\}, \quad \hat{N}_2 = \left\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}, \quad \hat{N}_3 = \left\{-\sqrt[3]{\frac{3}{5}}, 0, \sqrt[3]{\frac{3}{5}}\right\}, \dots$$

For any $x_0 \in N_k$,

$$|(u - \Pi_h u)'(x_0)| \leq Ch^k \|u\|_{k+2,1,K}.$$

For the two-dimensional linear triangular elements, the interpolation optimal stress points are the midpoints of the sides. For a uniform mesh one has

$$|\bar{\nabla}(u - \Pi_h u)(x_0)| \leq C|u|_{3,1,E}. \quad (2.6.3)$$

In the case of nearly uniform mesh

$$|\bar{\nabla}(u - \Pi_h u)(x_0)| \leq Ch^{2-2/q} \|u\|_{3,q,E} \quad (q > 2). \quad (2.6.4)$$

Further details can be found in the references on the superconvergence of finite element methods.

Just as in the case of finite element methods, we can also obtain superconvergence results for generalized difference methods, provided we manage to get the super interpolation weak estimates. In detail we have the following theorem.

Theorem 2.6.1 *Let u and u_h be solutions of the boundary value problem and its generalized difference scheme, respectively. Assume the bilinear form of the generalized difference scheme satisfies the following interpolation weak estimate: There exists $p \in [1, \infty]$ such that*

$$\begin{aligned} & |a(u - \Pi_h u, \Pi_h^* w_h)| \\ & \leq Ch^{k+1} \|u\|_{k+2,p} \|w_h\|_1, \quad \forall w_h \in U_h. \end{aligned} \quad (2.6.5)$$

Then one has

$$\|\Pi_h u - u_h\|_1 \leq Ch^{k+1} \|u\|_{k+2,p}. \quad (2.6.6)$$

Moreover, let N_k be the set of interpolation optimal stress points: For any $x_0 \in N_k$ there exists $q \in [1, \infty]$ such that

$$|\bar{\nabla}(u - \Pi_h u)(x_0)| \leq Ch^{k+1-\frac{N}{q}} \|u\|_{k+2,q,E}. \quad (2.6.7)$$

Then one has

$$\left[\frac{1}{r} \sum_{x_0 \in N_k} |\bar{\nabla}(u - u_h)(x_0)|^2 \right]^{1/2} \leq Ch^{k+1} \|u\|_{k+2,\infty}, \quad (2.6.8)$$

where r is the number of points in N_k . (Usually the number of super points in each element is fixed, so $r = O(h^{-N})$ for an N -dimensional problem.)

Proof It follows from the uniform U_h -ellipticity of the bilinear form and the weak estimate (2.6.5) that

$$\begin{aligned} & \|\Pi_h u - u_h\|_1^2 \\ & \leq Ca(\Pi_h u - u_h, \Pi_h^*(\Pi_h u - u_h)) \\ & = Ca(\Pi_h u - u, \Pi_h^*(\Pi_h u - u_h)) \\ & \leq Ch^{k+1}\|u\|_{k+2,p}\|\Pi_h u - u_h\|_1, \end{aligned}$$

which implies (2.6.6).

The inverse property of finite element methods (Theorem 1.1.13) leads to

$$|\bar{\nabla}(\Pi_h u - u_h)(x_0)| \leq Ch^{-N/2}\|\Pi_h u - u_h\|_{1,E}.$$

Noticing $r = O(h^{-N})$ and using (2.6.6), we have

$$\begin{aligned} & \left[\frac{1}{r} \sum_{x_0 \in N_k} |\bar{\nabla}(\Pi_h u - u_h)(x_0)|^2 \right]^{1/2} \\ & \leq C\|\Pi_h u - u_h\|_1 \leq Ch^{k+1}\|u\|_{k+2,p}. \end{aligned} \quad (2.6.9)$$

(2.6.7) gives

$$|\bar{\nabla}(u - \Pi_h u)(x_0)| \leq Ch^{k+1}\|u\|_{k+2,\infty}.$$

Thus

$$\left[\frac{1}{r} \sum_{x_0 \in N_k} |\bar{\nabla}(u - \Pi_h u)(x_0)|^2 \right]^{1/2} \leq Ch^{k+1}\|u\|_{k+2,\infty}. \quad (2.6.10)$$

Now (2.6.8) follows from (2.6.9) and (2.6.10). This completes the proof. $\square$

Remark If $p = q = 2$ in (2.6.5) and (2.6.7), then the right-hand side of (2.6.8) can be replaced by $Ch^{k+1}\|u\|_{k+2}$.

2.6.2 Superconvergence for linear element difference schemes

Let us consider the linear element difference scheme (2.2.6) for the two point boundary value problem (2.2.1). We first deduce a relevant interpolation weak estimate, then give a superconvergence result.

Theorem 2.6.2 *For the linear element difference scheme (2.2.6) approximating the two point boundary value problem (2.2.1), the following interpolation weak estimate holds:*

$$\begin{aligned} |a(u - \Pi_h u, \Pi_h^* w_h)| &\leq Ch^2 \|u\|_{3,p} \|w_h\|_{1,p'}, \\ \forall u \in W^{3,p}(I), w_h \in U_h; 1 \leq p, p' \leq +\infty; \frac{1}{p} + \frac{1}{p'} &= 1. \end{aligned} \quad (2.6.11)$$

Proof In §2.2 we have found that

$$a(u - \Pi_h u, \Pi_h^* w_h) = \sum_{j=1}^n p_{j-1/2} (u - \Pi_h u)'(x_{j-1/2}) [w_h(x_j) - w_h(x_{j-1})].$$

On $I_j = [x_{j-1}, x_j]$

$$(\Pi_h u)' = [u(x_j) - u(x_{j-1})]/h_j.$$

By the Taylor expansion with integral remainder we have

$$\begin{aligned} &u(x_j) - u(x_{j-1}) \\ &= u'(x_{j-1/2})h_j + \frac{1}{2!} \int_{x_{j-1/2}}^{x_j} u'''(x)(x_j - x)^2 dx \\ &\quad - \frac{1}{2!} \int_{x_{j-1/2}}^{x_{j-1}} u'''(x)(x_{j-1} - x)^2 dx. \end{aligned}$$

Thus

$$\begin{aligned} &(u - \Pi_h u)'(x_{j-1/2}) \\ &= -\frac{1}{2h_j} \left[\int_{x_{j-1/2}}^{x_j} u'''(x)(x_j - x)^2 dx + \int_{x_{j-1}}^{x_{j-1/2}} u'''(x)(x_{j-1} - x)^2 dx \right]. \end{aligned}$$

Noting

$$w_h(x_j) - w_h(x_{j-1}) = w_h'(x)h_j, \quad x \in I_j,$$

we have

$$|a(u - \Pi_h u, \Pi_h^* w_h)| \leq Ch^2 \sum_{j=1}^n |u|_{3,p,I_j} |w_h|_{1,p',I_j} \leq Ch^2 |u|_{3,p} |w_h|_{1,p'}.$$

This completes the proof. $\square$

Now, combining Theorems 2.6.1 and 2.6.2 leads to the following superconvergence result for the linear element difference scheme.

Theorem 2.6.3 *Let $u \in H_E^1(I)$ be the solution of the two point boundary value problem (2.2.1) and $u_h \in U_h$ of the linear element difference scheme (2.2.6). Assume in addition $u \in H^3(I)$. Then*

$$\|\Pi_h u - u_h\|_1 \leq Ch^2 \|u\|_3, \quad (2.6.12)$$

$$\left[\frac{1}{n} \sum_{j=1}^n |(u - u_h)'(x_{j-1/2})|^2 \right]^{1/2} \leq Ch^2 \|u\|_3. \quad (2.6.13)$$

2.6.3 Superconvergence for cubic element difference schemes

Consider the cubic element difference scheme (2.4.4) for the two point boundary value problem (2.4.1). First we give a lemma.

Lemma 2.6.1 *If $u \in H^5(I)$, then for $j = 1, 2, \dots, n$*

$$\begin{aligned} & (\Pi_h u)(x_{j-1/2}) \\ &= u(x_{j-1/2}) - \frac{1}{4!} u^{(4)}(x_{j-1/2}) \left(\frac{h}{2}\right)^4 \\ & \quad + \frac{1}{2 \cdot 4!} \left[\int_{x_{j-1/2}}^{x_j} u^{(5)}(x) (x_j - x)^4 dx \right. \\ & \quad \left. - \int_{x_{j-1}}^{x_{j-1/2}} u^{(5)}(x) (x_{j-1} - x)^4 dx \right] \\ & \quad - \frac{h}{8 \cdot 3!} \left[\int_{x_{j-1/2}}^{x_j} u^{(5)}(x) (x_j - x)^3 dx \right. \\ & \quad \left. + \int_{x_{j-1}}^{x_{j-1/2}} u^{(5)}(x) (x_{j-1} - x)^3 dx \right], \end{aligned} \quad (2.6.14)$$

$$\begin{aligned}
& (\Pi_h u)'(x_{j-1/2}) \\
&= u'(x_{j-1/2}) + \frac{3}{2 \cdot 4!h} \left[\int_{x_{j-1/2}}^{x_j} u^{(5)}(x)(x_j - x)^4 dx \right. \\
&\quad \left. + \int_{x_{j-1}}^{x_{j-1/2}} u^{(5)}(x)(x_{j-1} - x)^4 dx \right] \\
&\quad - \frac{1}{4 \cdot 3!} \left[\int_{x_{j-1/2}}^{x_j} u^{(5)}(x)(x_j - x)^3 dx \right. \\
&\quad \left. - \int_{x_{j-1}}^{x_{j-1/2}} u^{(5)}(x)(x_{j-1} - x)^3 dx \right].
\end{aligned} \tag{2.6.15}$$

Proof By (2.4.2) and (2.4.3) we have

$$\begin{aligned}
& (\Pi_h u)(x_{j-1/2}) \\
&= \frac{1}{2}[u(x_j) + u(x_{j-1})] - \frac{h}{8}[u'(x_j) - u'(x_{j-1})], \\
& (\Pi_h u)'(x_{j-1/2}) \\
&= \frac{3}{2h}[u(x_j) - u(x_{j-1})] - \frac{1}{4}[u'(x_j) + u'(x_{j-1})].
\end{aligned}$$

Then the Taylor expansion with integral remainders leads to (2.6.14) and (2.6.15). $\square$

Theorem 2.6.4 Let T_h be a uniform grid ($h_j = h$, $j = 1, 2, \dots, n$). Then the cubic element scheme (2.4.4) for the two point boundary value problem (2.4.1) satisfies the following interpolation weak estimate:

$$|a(u - \Pi_h u, \Pi_h^* w_h)| \leq Ch^4 \|u\|_5 \|w_h\|_1, \quad \forall u \in H^5(I), \quad w_h \in U_h. \tag{2.6.16}$$

Proof First we have.

$$\begin{aligned}
& a(u - \Pi_h u, \Pi_h^* w_h) \\
&= \sum_{j=1}^n (L(u - \Pi_h u), \psi_j^{(0)}) w_h(x_j) + \sum_{j=0}^n (L(u - \Pi_h u), \psi_j^{(1)}) w_h'(x_j).
\end{aligned} \tag{2.6.17}$$

Using (2.6.15) and the equivalent norm (2.4.14), we find that

$$\begin{aligned}
& \left| \sum_{j=1}^n (-[p(u - \Pi_h u)']', \psi_j^{(0)}) w_h(x_j) \right| \\
&= \left| \sum_{j=1}^n [p(x_{j-1/2})(u - \Pi_h u)'(x_{j-1/2}) \right. \\
&\quad \left. - p(x_{j+1/2})(u - \Pi_h u)'(x_{j+1/2})] w_h(x_j) \right| \\
&= \left| \sum_{j=1}^n p(x_{j-1/2})(u - \Pi_h u)'(x_{j-1/2}) [w_h(x_j) - w_h(x_{j-1})] \right| \\
&\leq Ch^4 |u|_5 |w_h|_1.
\end{aligned} \tag{2.6.18}$$

It follows from (2.6.14) that

$$|(u - \Pi_h u)(x_{j+1/2}) - (u - \Pi_h u)(x_{j-1/2})| \leq Ch^4 \int_{x_{j-1}}^{x_{j+1}} |u^{(5)}(x)| dx. \tag{2.6.19}$$

It is clear from (2.4.2) that the norm $\|w_h\|_0$ is equivalent to the discrete norm

$$\begin{aligned}
\|w\|_{0,h} = & \left\{ \sum_{j=1}^n h_j [(w_h(x_{j-1}))^2 + (w_h(x_j))^2 \right. \\
& \left. + (h_j w_h'(x_{j-1}))^2 + (h_j w_h'(x_j))^2] \right\}^{1/2}.
\end{aligned} \tag{2.6.20}$$

So we find

$$\begin{aligned}
& \left| \sum_{j=1}^n (r(u - \Pi_h u)', \psi_j^{(0)}) w_h(x_j) \right| \\
&= \left| \sum_{j=1}^n r(x_j) [(u - \Pi_h u)(x_{j+1/2}) - (u - \Pi_h u)(x_{j-1/2})] w_h(x_j) \right. \\
&\quad \left. + \sum_{j=1}^n \int_{x_{j-1/2}}^{x_{j+1/2}} [r(x) - r(x_j)] (u - \Pi_h u)' dx w_h(x_j) \right| \\
&\leq Ch^4 \|u\|_5 \|w_h\|_0.
\end{aligned} \tag{2.6.21}$$

It is obvious that

$$\begin{aligned}
& \left| \sum_{j=1}^n (q(u - \Pi_h u), \psi_j^{(0)}) w_h(x_j) \right| \\
& \leq C \sum_{j=1}^n \int_{x_{j-1/2}}^{x_{j+1/2}} |u - \Pi_h u| dx |w_h(x_j)| \\
& \leq Ch^4 \|u\|_4 \|w_h\|_0.
\end{aligned} \tag{2.6.22}$$

Similarly as in (2.6.21) we use (2.6.15) to obtain

$$\begin{aligned}
& \left| \sum_{j=0}^n (-[p(u - \Pi_h u)']', \psi_j^{(1)}) w_h'(x_j) \right| \\
& = \left| \sum_{j=0}^n \left[-\frac{h}{2} p(x_{j+1/2}) (u - \Pi_h u)'(x_{j+1/2}) \right. \right. \\
& \quad \left. \left. - \frac{h}{2} p(x_{j-1/2}) (u - \Pi_h u)'(x_{j-1/2}) \right. \right. \\
& \quad \left. \left. + \int_{x_{j-1/2}}^{x_{j+1/2}} p(u - \Pi_h u)' dx \right] w_h'(x_j) \right| \\
& \leq Ch^4 \|u\|_5 |w_h|_1.
\end{aligned} \tag{2.6.23}$$

It is an easy matter to deduce that

$$\begin{aligned}
& \left| \sum_{j=1}^n (r(u - \Pi_h u)' + q(u - \Pi_h u), \psi_j^{(1)}) w_h'(x_j) \right| \\
& = \left| \sum_{j=1}^n \int_{x_{j-1/2}}^{x_{j+1/2}} [r(u - \Pi_h u)' + q(u - \Pi_h u)](x - x_j) dx w_h'(x_j) \right| \\
& \leq Ch^4 |u|_4 |w_h|_1.
\end{aligned} \tag{2.6.24}$$

Now (2.6.16) follows from (2.6.17), (2.6.18) and (2.6.21)-(2.6.24). This completes the proof. $\square$

Theorems 2.6.4 and 2.6.1 imply the following superconvergence estimate.

Theorem 2.6.5 *Let u be the solution of the two boundary value problem (2.4.1) with $u \in H_E^1(I) \cap H^5(I)$, u_h of the cubic element difference scheme (2.4.4), and T_h a uniform grid. Then we have*

$$\|\Pi_h u - u_h\|_1 \leq Ch^4 \|u\|_5,$$

$$\left[\frac{1}{r} \sum |(u - u_h)'(x_0)|^2 \right]^{1/2} \leq Ch^4 \|u\|_5.$$

2.7 Generalized Difference Methods for a Fourth Order Equation

As an example of high order equations, let us consider the beam balance equation:

$$\begin{cases} Lu \equiv \frac{d^2}{dx^2} \left(p \frac{du^2}{dx^2} \right) = f(x), & a \leq x \leq b, & (2.7.1a) \\ u(a) = u(b) = 0, & & (2.7.1b) \\ u'(a) = u'(b) = 0, & & (2.7.1c) \end{cases}$$

where $p \geq p_{\min} > 0$, $p \in C^1(I)$, $f \in L^2(I)$. In this section we shall first derive a generalized difference scheme in terms of the Hermite cubic element, then give its error estimates.

2.7.1 Generalized difference equations

The variational problem in accordance with (2.7.1) is: Find $u \in U = H_0^2(I)$ such that

$$a(u, v) = (f, v), \quad \forall v \in U, \quad (2.7.2)$$

where

$$a(u, v) = \int_a^b p u'' v'' dx. \quad (2.7.3)$$

Discretize I as in §2.4, and take the trial and test spaces U_h and V_h as Hermite cubic element and piecewise linear function spaces, respectively. They and their derivatives are identically zero at the boundary nodes a and b .

The generalized difference scheme approximating (2.7.1) is: Find $u_h = \sum_{i=1}^{n-1} (u_i \phi_i^{(0)} + u'_i \phi_i^{(1)})$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad (2.7.4)$$

or equivalently

$$\begin{cases} a(u_h, \psi_j^{(0)}) = (f, \psi_j^{(0)}), & j = 1, 2, \dots, n-1, \\ a(u_h, \psi_j^{(1)}) = (f, \psi_j^{(1)}), & j = 1, 2, \dots, n-1. \end{cases} \quad (2.7.4a) \quad (2.7.4b)$$

As pointed out in §2.1.3, here we can explain $a(u, v)$ either as (2.7.3) in the sense of generalized functions, or as the following bilinear form by piecewise integrating (Lu, v) by parts

$$\begin{aligned} a(u, v) &= \sum_{j=0}^n \int_{x_{j-1/2}}^{x_{j+1/2}} (pu'')' v dx \\ &= \sum_{j=0}^n \left[(pu'')' v \Big|_{x_{j-1/2}^+}^{x_{j+1/2}^-} - \int_{x_{j-1/2}}^{x_{j+1/2}} (pu'')' v' dx \right] \\ &= \sum_{j=0}^n \left[(pu'')' v \Big|_{x_{j-1/2}^+}^{x_{j+1/2}^-} - pu'' v' \Big|_{x_{j-1/2}^+}^{x_{j+1/2}^-} \right], \\ &\quad v \in V_h, \end{aligned} \quad (2.7.5)$$

where we make the convention that $x_{-1/2} = x_0$ and $x_{n+1/2} = x_n$.

Next we calculate (2.7.4a) and (2.7.4b).

$$\begin{aligned} a(u_h, \psi_j^{(0)}) &= \int_a^b pu_h'' \psi_j^{(0)'} dx \\ &= \int_a^b pu_h'' [\delta'(x - x_{j-1/2}) - \delta'(x - x_{j+1/2})] dx \\ &= -(pu_h'')'_{j-1/2} + (pu_h'')'_{j+1/2} \\ &= -(p'u_h'')_{j-1/2} - (pu_h''')_{j-1/2} \\ &\quad + (p'u_h'')_{j+1/2} + (pu_h''')_{j+1/2}. \end{aligned} \quad (2.7.6)$$

$$\begin{aligned}
a(u_h, \psi_j^{(1)}) &= \int_a^b p u_h'' \psi_j^{(1)''} dx \\
&= \int_a^b p u_h'' \left[\delta(x - x_{j-1/2}) - \delta(x - x_{j+1/2}) \right. \\
&\quad \left. - \frac{h_j}{2} \delta'(x - x_{j-1/2}) - \frac{h_{j+1}}{2} \delta'(x - x_{j+1/2}) \right] dx \\
&= (p u_h'')_{j-1/2} - (p u_h'')_{j+1/2} + \frac{h_j}{2} (p u_h'')'_{j-1/2} + \frac{h_{j+1}}{2} (p u_h'')'_{j+1/2}.
\end{aligned} \tag{2.7.7}$$

For sake of brevity we shall write $u_h(x_j) = (u_h)_j = u_j$ etc. when there is no possible confusion. By (2.4.3) we have on $[x_{j-1}, x_j]$

$$\begin{aligned}
u_h'' &= (12\xi - 6)h_j^{-2}u_{j-1} - (12\xi - 6)h_j^{-2}u_j \\
&\quad + (6\xi - 4)h_j^{-1}u'_{j-1} + (6\xi - 2)h_j^{-1}u'_j \\
&= [\xi, 1] \begin{bmatrix} 6 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} \{u'_{j-1} + u'_j - 2h_j^{-1}(u_j - u_{j-1})\}/h_j \\ (u'_j - u'_{j-1})/h_j \end{bmatrix},
\end{aligned} \tag{2.7.8}$$

$$u_h''' = 6 \frac{u'_{j-1} + u'_j - 2h_j^{-1}(u_j - u_{j-1})}{h_j^2}. \tag{2.7.9}$$

Substituting (2.7.8) and (2.7.9) into (2.7.6) and (2.7.7) yields

$$\begin{aligned}
&a(u_h, \psi_j^{(0)}) \\
&= -12p_{j-1/2}h_j^{-3}u_{j-1} + 12(p_{j-1/2}h_j^{-3} + p_{j+1/2}h_{j+1}^{-3})u_j \\
&\quad -12p_{j+1/2}h_{j+1}^{-3}u_{j+1} + (-6p_{j-1/2}h_j^{-2} + p'_{j-1/2}h_j^{-1})u'_{j-1} \\
&\quad + (6p_{j+1/2}h_{j+1}^{-2} + p'_{j+1/2}h_{j+1}^{-1})u'_{j+1} \\
&\quad + (-6p_{j-1/2}h_j^{-2} - p'_{j-1/2}h_j^{-1} + 6p_{j+1/2}h_{j+1}^{-2} - p'_{j+1/2}h_{j+1}^{-1})u'_j, \\
&\tag{2.7.10} \\
&a(u_h, \psi_j^{(1)}) \\
&= 6p_{j-1/2}h_j^{-2}u_{j-1} + (-6p_{j-1/2}h_j^{-2} + 6p_{j+1/2}h_{j+1}^{-2})u_j \\
&\quad -6p_{j+1/2}h_{j+1}^{-2}u_{j+1}
\end{aligned}$$

$$\begin{aligned}
& + (2p_{j-1/2}h_j^{-1} - \frac{1}{2}p'_{j-1/2})u'_{j-1} + (2p_{j+1/2}h_{j+1}^{-1} + \frac{1}{2}p'_{j+1/2})u'_{j+1} \\
& + (4p_{j-1/2}h_j^{-1} + \frac{1}{2}p'_{j-1/2} + 4p_{j+1/2}h_{j+1}^{-1} - \frac{1}{2}p'_{j+1/2})u'_j,
\end{aligned} \tag{2.7.11}$$

where $u_0 = u'_0 = u_n = u'_n = 0$.

2.7.2 Positive definiteness of $a(u_h, \Pi_h^* u_h)$

It is well-known that the seminorm $|\cdot|_2$ is equivalent to the full norm $\|\cdot\|_2$ on the space H_0^2 . Now we introduce an equivalent discrete norm. Motivated by (2.7.8), we define

$$\begin{aligned}
|u_h|_{2,h} = & \left\{ \sum_{j=1}^n h_j \left[\left(\frac{u'_{j-1} + u'_j - 2h_j^{-1}(u_j - u_{j-1})}{h_j} \right)^2 \right. \right. \\
& \left. \left. + \left(\frac{u'_j - u'_{j-1}}{h_j} \right)^2 \right] \right\}^{1/2}, \quad u_h \in U_h.
\end{aligned} \tag{2.7.12}$$

The following lemma can be easily proved similarly as Theorem 2.3.1.

Lemma 2.7.1 The norms $|\cdot|_{2,h}$ and $|\cdot|_2$ are equivalent on U_h , i.e., there exist constants c_1 and c_2 independent of the subspace U_h such that

$$c_1|u_h|_{2,h} \leq |u_h|_2 \leq c_2|u_h|_{2,h}, \quad \forall u_h \in U_h. \tag{2.7.13}$$

Using Lemma 2.7.1 one can show the following uniform ellipticity theorem.

Theorem 2.7.1 $a(u_h, \Pi_h^* u_h)$ is positive definite for sufficiently small h , i.e., there exists a positive constant α independent of U_h such that

$$a(u_h, \Pi_h^* u_h) \geq \alpha|u_h|_2^2, \quad \forall u_h \in U_h. \tag{2.7.14}$$

Proof By (2.7.6) and (2.7.7) we find that

$$\begin{aligned}
& a(u_h, \Pi_h^* u_h) \\
& = \sum_{j=1}^n [a(u_h, \psi_j^{(0)})u_j + a(u_h, \psi_j^{(1)})u'_j]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^n [(p'_{j+1/2} u''_{j+1/2} - p'_{j-1/2} u''_{j-1/2}) u_j \\
&\quad + (p_{j+1/2} u'''_{j+1/2} - p_{j-1/2} u'''_{j-1/2}) u_j \\
&\quad - (p_{j+1/2} u''_{j+1/2} - p_{j-1/2} u''_{j-1/2}) u'_j \\
&\quad + \frac{1}{2} (h_{j+1} p'_{j+1/2} u''_{j+1/2} + h_j p'_{j-1/2} u''_{j-1/2}) u'_j \\
&\quad + \frac{1}{2} (h_{j+1} p_{j+1/2} u'''_{j+1/2} + h_j p_{j-1/2} u'''_{j-1/2}) u'_j] \\
&= \sum_{j=1}^n [p'_{j-1/2} u''_{j-1/2} (u_{j-1} - u_j) + p_{j-1/2} u'''_{j-1/2} (u_{j-1} - u_j) \\
&\quad + p_{j-1/2} u''_{j-1/2} (u'_j - u'_{j-1}) + \frac{1}{2} h_j p'_{j-1/2} u''_{j-1/2} (u'_j + u'_{j-1}) \\
&\quad + \frac{1}{2} h_j p_{j-1/2} u'''_{j-1/2} (u'_j + u'_{j-1})] \\
&= \sum_{j=1}^n [\frac{1}{2} h_j p'_{j-1/2} u''_{j-1/2} (u'_j + u'_{j-1} - 2h_j^{-1} (u_j - u_{j-1})) \\
&\quad + \frac{1}{2} h_j p_{j-1/2} u'''_{j-1/2} (u'_j + u'_{j-1} - 2h_j^{-1} (u_j - u_{j-1})) \\
&\quad + p_{j-1/2} u''_{j-1/2} (u'_j - u'_{j-1})].
\end{aligned} \tag{2.7.15}$$

By (2.7.8) one has

$$u''_{j-1/2} = \frac{u'_j - u'_{j-1}}{h_j}. \tag{2.7.16}$$

Finally we use (2.7.16), (2.7.9) and (2.7.15) to conclude that

$$\begin{aligned}
&a(u_h, \Pi_h^* u_h) \\
&= \sum_{j=1}^n \left[\frac{1}{2} h_j^2 p'_{j-1/2} \frac{u'_j - u'_{j-1}}{h_j} \frac{u'_j + u'_{j-1} - 2h_j^{-1} (u_j - u_{j-1})}{h_j} \right. \\
&\quad \left. + 3p_{j-1/2} h_j \left(\frac{u'_j + u'_{j-1} - 2h_j^{-1} (u_j - u_{j-1})}{h_j} \right)^2 \right]
\end{aligned}$$

$$\begin{aligned}
& + p_{j-1/2} h_j \left(\frac{u'_j - u'_{j-1}}{h_j} \right)^2 \Big] \\
& \geq p_{\min} |u_h|_{2,h}^2 - \frac{1}{4} h |u_h|_{2,h}^2 \max_{x \in I} |p'(x)|.
\end{aligned}$$

This leads to the desired result. $\square$

We remark that Theorem 2.7.1 implies the existence, uniqueness, and stability of the solution of the generalized difference scheme (2.7.4).

2.7.3 Convergence order estimates

Theorem 2.7.2 *Let u be the solution to (2.7.1) satisfying $u \in H_0^2(I) \cap H^4(I)$ and $u_h \in U_h$ to the generalized difference scheme (2.7.4). Then the following error estimate holds for sufficiently small h*

$$|u - u_h|_2 \leq C h^2 |u|_4. \quad (2.7.17)$$

Proof Clearly we have

$$a(u - u_h, \Pi_h^* w_h) = 0, \quad \forall w_h \in U_h. \quad (2.7.18)$$

By Theorem 2.7.1

$$\alpha |u_h - \Pi_h u|_2^2 \leq a(u_h - \Pi_h u, \Pi_h^*(u_h - \Pi_h u)) = a(u - \Pi_h u, \Pi_h^*(u_h - \Pi_h u)).$$

Consequently

$$|u_h - \Pi_h u|_2 \leq C \sup_{w_h \in U_h} \frac{|a(u - \Pi_h u, \Pi_h^* w_h)|}{|w_h|_2}. \quad (2.7.19)$$

Write $e_h = u - \Pi_h u$ and $e_h(x_{j-1/2}) = e_{j-1/2}$. Then, similar to (2.7.15) we have

$$\begin{aligned}
& a(e_h, \Pi_h^* w_h) \\
& = \sum_{j=1}^n \left[\frac{1}{2} p'_{j-1/2} h_j e''_{j-1/2} (w'_{j-1} + w'_j - 2h_j^{-1} (w_j - w_{j-1})) \right. \\
& \quad + \frac{1}{2} p_{j-1/2} h_j e'''_{j-1/2} (w'_{j-1} + w'_j - 2h_j^{-1} (w_j - w_{j-1})) \\
& \quad \left. + p_{j-1/2} e''_{j-1/2} (w'_j - w'_{j-1}) \right].
\end{aligned} \quad (2.7.20)$$

By the Cauchy inequality we have

$$\begin{aligned}
 & a(e_h, \Pi_h^* w_h) \\
 & \leq C \left\{ \sum_{j=1}^n [(e''_{j-1/2})^2 h_j^3 + (e'''_{j-1/2})^2 h_j^3 + (e''_{j-1/2})^2 h_j] \right\}^{1/2} \\
 & \quad \cdot \left\{ \sum_{j=1}^n h_j \left[\left(\frac{w'_{j-1} + w'_j - 2h_j^{-1}(w_j - w_{j-1})}{h_j} \right)^2 + \left(\frac{w'_j - w'_{j-1}}{h_j} \right)^2 \right] \right\}^{1/2}.
 \end{aligned} \tag{2.7.21}$$

By the interpolation condition and Rolle theorem we know that $e''_h = (u - \Pi_h u)''$ has two roots ξ_1, ξ_2 , and e'''_h has one root η in (x_{j-1}, x_j) . Hence,

$$e'''_h(x) = \int_{\eta}^x e_h^{(4)} dx = \int_{\eta}^x u^{(4)} dx,$$

$$(e'''_h(x))^2 \leq h \int_{x_{j-1}}^{x_j} |u^{(4)}|^2 dx, \quad x \in I_j,$$

$$e''_h(x) = \int_{\xi_1}^x e'''_h(x) dx,$$

$$(e''_h(x))^2 \leq h^3 \int_{x_{j-1}}^{x_j} |u^{(4)}|^2 dx, \quad x \in I_j.$$

Substituting these estimates into (2.7.21) and using Lemma 2.7.1 we have

$$|a(e_h, \Pi_h^* w_h)| \leq Ch^2 |u|_4 |w_h|_2. \tag{2.7.22}$$

This together with (2.7.19) results in

$$|u_h - \Pi_h u|_2 \leq Ch^2 |u|_4. \tag{2.7.23}$$

Finally, the desired result (2.7.17) follows from (2.7.23) and the interpolation property

$$|u - \Pi_h u|_2 \leq Ch^2 |u|_4. \quad \square$$

2.7.4 Numerical examples

The five point difference method (FDM), the cubic finite element method (FEM) and the cubic generalized difference method (GDM) discussed in this section are used to solve the following model problem:

$$\begin{cases} u^{(4)}(x) = x - \frac{1}{2}, & 0 < x < 1, \\ u(0) = u(1) = 0, & u'(0) = u'(1) = 0. \end{cases}$$

Set the step length $h = 0.1$. The results of the three methods and the true solution (TS) $u = \frac{1}{61}x^2(x-1)^2(x-\frac{1}{2})$ are given in Table 2.7.1 below. We see that (GDM) is much more accurate than (FDM) and is nearly as accurate as (FEM), while we recall that (GDM) needs much less computational effort than (FEM).

The upper members of each pair in Table 2.7.1 stand for function values and the lower ones derivatives.

Table 2.7.1 Numerical results

x	FDM	FEM	GDM	TS
0.1	-0.00003882	-0.00002700	-0.00002710	-0.00002700
		-0.00041250	-0.00041243	-0.00041250
0.2	-0.00007976	-0.00006400	-0.00006401	-0.00006400
		-0.00026667	-0.00026653	-0.00026667
0.3	-0.00008729	-0.00007350	-0.00007351	-0.00007350
		0.00087500	0.00087675	0.00087500
0.4	-0.00005588	-0.00004800	-0.00004807	-0.00004800
		0.00040000	0.00040020	0.00040000
0.5	0.00000000	0.00000000	0.00000000	0.00000000
		0.00052083	0.00052104	0.00052083

Bibliography and Comments

[A-25] is the earliest paper on the generalized difference methods. (Before that, at the Conference of China Mathematical Society at Chengdu in 1978, Ronghua Li gave a talk on the way to construct

the generalized difference schemes.) The original motive is to generalize the usual finite difference method (including the difference method on irregular networks) such as to possess the advantages of both finite element and finite difference methods, in particular to enjoy the same convergence order as finite element methods and less computational effort as finite difference methods, while including the usual difference methods as its special case. The key step is to use the general terms of the Taylor series as the basis functions of the test function space so as to gain the computational simplicity at the price of the loss of global smoothness. The error estimates for generalized Galerkin methods by Babuska (see (1.2.42) in Chapter 1) give an inspiration to the possible convergence orders to be reached, but it fails to provide a rigorous and practical approach for further development. The references [A-25,30] and [A-53] provide a framework for the error estimates of the generalized difference methods. It is indicated by theoretical analysis as well as numerical experiments that the generalized difference methods indeed have the same convergence order as the finite element methods. The results in §§2.1, 2.2 and 2.4 come out of [A-25] and [A-53]. For the generalization of the quadratic element difference scheme to two-dimensional problems, see [A-41] and §3.4 of this book. Another form of quadratic element difference scheme is constructed in [A-54]. There are some difficulties in using the Nitsche argument to estimate the L^2 -error when piecewise constant functions are adopted as test functions: A higher smoothness, compared with the finite element methods, is required to obtain the optimal order estimates. It is an open question whether this result could be improved (cf. [A-10]). The estimates in L^2 and maximum norms for quadratic element difference schemes are rather tedious and are left for interested readers.

[A-36] generalizes some superconvergence results of finite element methods to generalized difference methods, resulting in certain superconvergence estimates for the cubic element difference scheme for two point boundary value problems. Superconvergence results for linear element difference schemes are given in §2.6. The superconvergence of quadratic element difference schemes remains to be tackled.

When proceeding from second order to higher order differential equations, the construction of the generalized difference method and its theoretical analysis will encounter new difficulties. The results on the generalized difference method for a beam balance problem in §2.7 of this chapter belong to [A-35]. A class of nonconforming generalized difference methods are presented in Chapter 4 for a two-dimensional high order differential equation.

Chapter 3

SECOND ORDER ELLIPTIC EQUATIONS

3.1 Introduction

The research on the difference methods on irregular meshes over a plane region can be traced back at least to MacNeal [B-65]. But it did not develop very much theoretically or practically at that time. In the last twenty odd years, there has appeared increasingly more research on the theories and applications of the difference schemes on irregular meshes. These methods are also called in the references the finite control volume methods, or the finite volume methods. (See the corresponding references at the end of the book.) The generalized difference methods can be viewed as a generalization of the difference methods on irregular networks by absorbing the idea of the finite element methods.

Let Ω be a bounded region with a piecewise smooth boundary $\partial\Omega$ on the (x, y) plane. Consider the first boundary value problem of the second order elliptic partial differential equation:

$$\begin{cases} Au \equiv -\left[\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) \right. & (3.1.1a) \\ \quad \left. + \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right)\right] + qu = f, (x, y) \in \Omega & (3.1.1b) \\ u|_{\partial\Omega} = 0, \end{cases}$$

where the coefficients $a_{ij}(x, y)$ ($i, j = 1, 2$) and $q(x, y)$ are sufficiently smooth functions satisfying the elliptic condition: There exists a constant $r > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x, y)\xi_i\xi_j \geq r \sum_{i=1}^n \xi_i^2, \quad q(x, y) \geq 0$$

holds for any real vector $(\xi_1, \xi_2) \in R^2$ and $(x, y) \in \bar{\Omega}$. We also require $f \in L^2(\Omega)$.

The corresponding variational problem for (3.1.1) is: Find $u \in U = H_0^1(\Omega)$ satisfying

$$a(u, v) = (f, v), \quad \forall v \in U, \quad (3.1.2)$$

where

$$a(u, v) = \int_{\Omega} \left[\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial x} + \left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial y} + quv \right] dx dy, \quad (3.1.3a)$$

$$(f, v) = \int_{\Omega} f v dx dy. \quad (3.1.3b)$$

The solution to (3.1.2) is called the generalized solution or the weak solution of (3.1.1).

Let U_h and V_h respectively be the suitably chosen trial and test spaces with the same finite dimension. The generalized Galerkin method is: Find $u_h \in U_h$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \quad (3.1.4)$$

If $U_h = V_h \subset U$, then (3.1.4) becomes the standard Galerkin method. Usually the (conforming) finite element methods use the interpolation of spline functions to construct the piecewise polynomial space $U_h = V_h \subset U$. Breach of the inclusion relationship, namely

$U_h = V_h \not\subset U$, leads to nonconforming methods. As mentioned in §2.1.3, generalized difference methods choose $U_h \subset U$ like finite element methods, but choose $V_h (\neq U_h)$ as lower order piecewise polynomial spaces.

It should be pointed out that for the generalized difference methods, we always have $V_h \not\subset U$. As in the case of nonconforming finite element methods, this is due to the loss of continuity of the functions in V_h on the boundary of two neighbouring elements. So the bilinear form $a(u, v)$ must be revised accordingly. For nonconforming finite element methods, the idea is to write the integral on the whole region as a sum of the integrals on every element K , so (3.1.3a) is rewritten as

$$a(u, v) = \sum_K \int_K \left[\left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial x} + \left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial y} + quv \right] dx dy. \quad (3.1.5)$$

Now $a(u, v)$ is well-defined on $U_h \times V_h$. For the generalized difference methods, we place a dual grid and interpret (3.1.3) in the sense of generalized functions, i.e., δ -functions on the boundary of neighbouring dual elements. Or equivalently, we take $a(u, v)$ as the bilinear form resulting from the piecewise integrations in parts on the dual elements K^* :

$$\int_{\Omega} Au \cdot v dx dy = \sum_{K^*} \int_{K^*} Au \cdot v dx dy.$$

So we have

$$\begin{aligned} & a(u, v) \\ &= \sum_{K^*} \int_{K^*} \left[\left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial x} \right. \\ & \quad \left. + \left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial y} + quv \right] dx dy \\ & \quad - \sum_{K^*} \int_{\partial K^*} \left[\left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) v dy - \left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) v dx \right], \end{aligned} \quad (3.1.6)$$

where $\int_{\partial K^*}$ denotes the line integrals, in the counterclockwise direction, on the boundary ∂K^* of the dual element.

Now (3.1.4) is an algebraic system for the approximate solutions of u and its derivatives. Different choices of U_h and V_h lead to different schemes. In particular, if we take V_h as the piecewise constant function space with the characteristic functions of the dual elements K^* as the basis functions, then the above method becomes the integral interpolation method based on the integral conservation law (the balance equation)

$$\begin{aligned} & \int_{K^*} A u dx dy \\ &= - \int_{\partial K^*} \left[\left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) dy \right. \\ & \quad \left. - \left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) dx \right] + \int_{K^*} q u dx dy \\ &= \int_{K^*} f dx dy. \end{aligned}$$

So the generalized difference method is a significant generalization of the finite difference method.

In the following sections different generalized difference schemes will be deduced and discussed by introducing different U_h and V_h .

3.2 Generalized Difference Methods on Triangular Meshes

3.2.1 Trial and test function spaces

The construction of the trial and test spaces is always related to a certain mesh decomposition. Suppose Ω is a polygonal region with boundary $\partial\Omega$. Divide Ω into a sum of finite number of small triangles such that they have no overlapping internal region; that a vertex of any triangle does not belong to a side of any other triangle; and that each vertex of $\partial\Omega$ is a vertex of a small triangle. Each triangle is called an element and the vertexes of the triangles are called nodes. Two elements are adjacent, if they share a common side. Two nodes are

adjacent if they are the endpoints of the same side. All the elements K constitute a triangulation of $\bar{\Omega}$, denoted by T_h , where h is the maximum length of all the sides.

The following definition will be used throughout this book.

Definition 3.2.1 We use $\overline{PQ}$ to denote the line segment with endpoints P and Q on the plane, which may bear a direction from P to Q when, e.g., it is a path of line integral. We also identify $\overline{PQ}$ with the corresponding vector of R^2 in the usual sense. Its length is denoted by $|\overline{PQ}|$.

Now we construct a dual decomposition T_h^* related to T_h . Let P_0 be a node of a triangle, P_i ($i = 1, 2, \dots, 6$) the adjacent nodes of P_0 , and M_i the midpoint of $\overline{P_0P_i}$ (cf. Fig. 3.2.1). Choose a point Q_i in an element $\triangle P_0P_iP_{i+1}$ ($P_7 = P_1$) and connect successively $M_1, Q_1, \dots, M_6, Q_6, M_1$ to form a polygonal region $K_{P_0}^*$, called a dual element. The modification of the definition is obvious when P_0 is on the boundary. All the dual elements constitute a new decomposition, called a dual decomposition (or a dual grid). Q_i is called a node of the dual decomposition. The following two dual decompositions are most important for the triangulation T_h :

(1) Barycenter dual decomposition. Take the barycenter Q_i of the triangle $\triangle P_0P_iP_{i+1}$ as the node of the dual decomposition, as shown in Fig. 3.2.1.

(2) Circumcenter dual decomposition. Assume that the interior angles of any element of T_h are not greater than 90° . Then, take the circumcenter Q_i of the element $\triangle P_0P_iP_{i+1}$ as the node of the dual decomposition. Now $\overline{Q_iQ_{i+1}}$ is the perpendicular bisector of $\overline{P_0P_{i+1}}$, cf. Fig. 3.2.2.

In the sequel we denote by $\bar{\Omega}_h$ the set of the nodes of the decomposition T_h , $\Omega_h = \bar{\Omega}_h \setminus \partial\Omega$ the set of the interior nodes, and Ω_h^* the set of the nodes of the dual decomposition T_h^* . For $Q \in \Omega_h^*$, K_Q denotes the triangular element containing Q . Let S_{K_Q} (or S_Q) and $S_{P_0}^*$ be the areas of the triangular element K_Q and the dual element $K_{P_0}^*$ respectively. It is easy to check that if T_h and T_h^* are quasi-uniform (cf. Definition 1.1.10), then there exist constant $c_1, c_2, c_3 > 0$ independent

of h such that

$$c_1 h^2 \leq S_Q \leq h^2, \quad Q \in \Omega_h^*, \quad (3.2.1a)$$

$$c_2 h^2 \leq S_{P_0}^* \leq c_3 h^2, \quad P_0 \in \bar{\Omega}_h. \quad (3.2.1b)$$

It can be readily shown that (3.2.1a) is actually a necessary and sufficient condition for the triangulation T_h to be quasi-uniform. Besides, for barycenter and circumcenter dual decompositions, (3.2.1b) can be deduced from (3.2.1a). In the sequel we always assume that the decomposition is quasi-uniform.

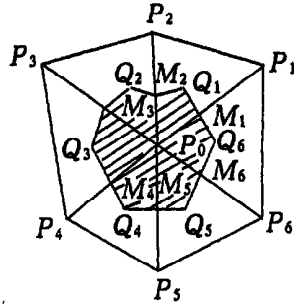


Fig. 3.2.1

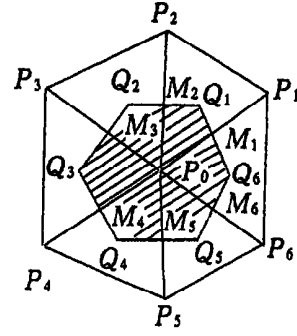


Fig. 3.2.2

The trial function space U_h is chosen as the linear element space related to T_h . So U_h is the set of all the functions u_h satisfying the following conditions:

- (i) $u_h \in C(\bar{\Omega})$, $u_h|_{\partial\Omega} = 0$;
- (ii) $u_h|_K \in \mathcal{P}_1$, namely u_h is a linear function of x and y on each triangular element $K \in T_h$, determined solely by its values on the three vertexes.

It is obvious that $U_h \subset U = H_0^1(\Omega)$.

Let $K = \triangle P_i P_j P_k$ be any triangular element and $P(x, y)$ a point in the element (cf. Fig. 3.2.3). Introduce the area coordinates

$(\lambda_i, \lambda_j, \lambda_k)$,

$$\lambda_i = \frac{S_i}{S} = \frac{1}{2S} \begin{vmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}, \quad (3.2.2a)$$

$$\lambda_j = \frac{S_j}{S} = \frac{1}{2S} \begin{vmatrix} 1 & x_i & y_i \\ 1 & x & y \\ 1 & x_k & y_k \end{vmatrix}, \quad (3.2.2b)$$

$$\lambda_k = \frac{S_k}{S} = \frac{1}{2S} \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x & y \end{vmatrix}, \quad (3.2.2c)$$

where S_i, S_j, S_k and S are the areas of $\triangle PP_jP_k, \triangle P_iPP_k, \triangle P_iP_jP$, and $\triangle P_iP_jP_k$, respectively. The mapping (3.2.2) maps $\triangle P_iP_jP_k$ onto a reference element $\hat{K}$ with vertexes $\hat{P}_i(0, 0), \hat{P}_j(1, 0)$ and $\hat{P}_k(0, 1)$ on the (λ_j, λ_k) plane (cf. Fig. 3.2.4).

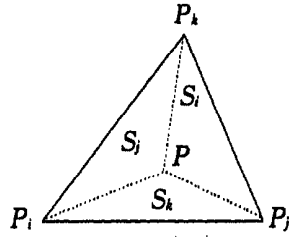


Fig. 3.2.3

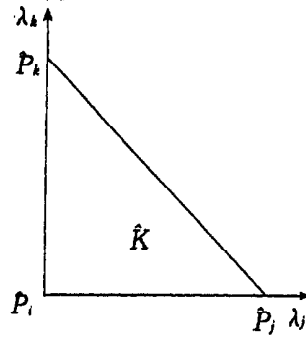


Fig. 3.2.4

The area coordinates and the orthogonal coordinates have the following relationship:

$$x = x_i\lambda_i + x_j\lambda_j + x_k\lambda_k, \quad (3.2.3a)$$

$$y = y_i\lambda_i + y_j\lambda_j + y_k\lambda_k, \quad (3.2.3b)$$

$$\lambda_i + \lambda_j + \lambda_k = 1. \quad (3.2.4)$$

It is easy to deduce that on the element K

$$u_h = u_i \lambda_i + u_j \lambda_j + u_k \lambda_k = u_i + (u_j - u_i) \lambda_j + (u_k - u_i) \lambda_k, \quad (3.2.5)$$

$$\begin{aligned} \frac{\partial u_h}{\partial x} &= \frac{1}{2S} \left[\frac{\partial u_h}{\partial \lambda_j} (y_k - y_i) + \frac{\partial u_h}{\partial \lambda_k} (y_i - y_j) \right] \\ &= \frac{1}{2S} [u_i (y_j - y_k) + u_j (y_k - y_i) + u_k (y_i - y_j)], \end{aligned} \quad (3.2.6)$$

$$\begin{aligned} \frac{\partial u_h}{\partial y} &= \frac{1}{2S} \left[\frac{\partial u_h}{\partial \lambda_j} (x_i - x_k) + \frac{\partial u_h}{\partial \lambda_k} (x_j - x_i) \right] \\ &= \frac{1}{2S} [u_i (x_k - x_j) + u_j (x_i - x_k) + u_k (x_j - x_i)], \end{aligned} \quad (3.2.7)$$

where and in the sequel, when there is no danger of confusion, we write in short $u_i = u_h(x_i, y_i)$ etc.

For $u \in U = H_0^1(\Omega)$, let $\Pi_h u$ be the interpolation projection of u onto the trial function space U_h . By the interpolation theory of Sobolev spaces we have, if $u \in H^2(\Omega)$, that

$$|u - \Pi_h u|_m \leq Ch^{2-m} |u|_2, \quad m = 0, 1, 2. \quad (3.2.8)$$

The test space V_h is chosen as the piecewise constant function space with respect to T_h^* , spanned by the following basis functions: For any point $P_0 \in \Omega_k$

$$\phi_{P_0}(P) = \begin{cases} 1, & P \in K_{P_0}^*, \\ 0, & \text{elsewhere.} \end{cases} \quad (3.2.9)$$

For any $v_h \in V_h$

$$v_h = \sum_{P_0 \in \Omega_h} v_h(P_0) \phi_{P_0}. \quad (3.2.10)$$

For $w \in U$, let $\Pi_h^* w$ be the interpolation projection of w onto the test space V_h :

$$\Pi_h^* w = \sum_{P_0 \in \Omega_h} w(P_0) \phi_{P_0}. \quad (3.2.11)$$

By the interpolation theory we have

$$|w - \Pi_h^* w|_0 \leq Ch |w|_1. \quad (3.2.12)$$

3.2.2 Generalized difference equation

Choose the trial function space U_h and the test function space V_h as above, then the generalized difference scheme is: Find $u_h \in U_h$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad (3.2.13)$$

or equivalently

$$a(u_h, \phi_{P_0}) = (f, \phi_{P_0}), \quad \forall P_0 \in \Omega_h, \quad (3.2.13)'$$

where

$$a(u_h, v_h) = \sum_{P_0 \in \Omega_h} v_h(P_0) a(u_h, \phi_{P_0}), \quad (3.2.14a)$$

$$\begin{aligned} & a(u_h, \phi_{P_0}) \\ &= - \int_{\partial K_{P_0}^*} [W_h^{(1)} \cos \langle n, x \rangle + W_h^{(2)} \cos \langle n, y \rangle] ds + \int_{K_{P_0}^*} q u_h dx dy \\ &= - \int_{\partial K_{P_0}^*} W_h^{(1)} dy + \int_{\partial K_{P_0}^*} W_h^{(2)} dx + \int_{K_{P_0}^*} q u_h dx dy, \end{aligned} \quad (3.2.14b)$$

where n is the unit outer normal vector and

$$W_h^{(1)} = a_{11} \frac{\partial u_h}{\partial x} + a_{12} \frac{\partial u_h}{\partial y}, \quad W_h^{(2)} = a_{21} \frac{\partial u_h}{\partial x} + a_{22} \frac{\partial u_h}{\partial y}.$$

Since ϕ_{P_0} is taken as the characteristic function of $K_{P_0}^*$, (3.2.13)' is in fact an integral conversation law (i.e. a balance equation) on $K_{P_0}^*$

$$\int_{K_{P_0}^*} A u dx dy = \int_{K_{P_0}^*} f dx dy.$$

We integrate in parts the left-hand side and then replace u by u_h , that is, we use the piecewise linear interpolation of the solution u .

As in Figs. 3.2.1 or 3.2.2, we employ different numerical integra-

tion formulas to approximate (3.2.14). For instance,

$$\begin{aligned}
& - \int_{Q_1 M_2 Q_2} W_h^{(1)} dy + \int_{Q_1 M_2 Q_2} W_h^{(2)} dx \\
& \doteq -[W_h^{(1)}]_{M_2} (y_{Q_2} - y_{Q_1}) + [W_h^{(2)}]_{M_2} (x_{Q_2} - x_{Q_1}) \\
& = - \left[a_{11}(M_2) \frac{u_{P_2} - u_{P_0}}{x_{P_2} - x_{P_0}} + a_{12}(M_2) \frac{u_{P_2} - u_{P_0}}{y_{P_2} - y_{P_0}} \right] (y_{Q_2} - y_{Q_1}) \\
& \quad + \left[a_{21}(M_2) \frac{u_{P_2} - u_{P_0}}{x_{P_2} - x_{P_0}} + a_{22}(M_2) \frac{u_{P_2} - u_{P_0}}{y_{P_2} - y_{P_0}} \right] (x_{Q_2} - x_{Q_1})
\end{aligned} \tag{3.2.15}$$

or

$$\begin{aligned}
& - \int_{M_1 Q_1 M_2} W_h^{(1)} dy + \int_{M_1 Q_1 M_2} W_h^{(2)} dx \\
& \doteq -[W_h^{(1)}]_{Q_1} (y_{M_2} - y_{M_1}) + [W_h^{(2)}]_{Q_1} (x_{M_2} - x_{M_1})
\end{aligned} \tag{3.2.16}$$

will lead to different conservative difference equations.

Next we take the Poisson equation

$$-\Delta u = f$$

as an example to study in detail the generalized difference scheme (3.2.13). Now

$$\begin{aligned}
a(u_h, \phi_{P_0}) &= - \int_{\partial K_{P_0}^*} \frac{\partial u_h}{\partial n} ds = - \int_{\partial K_{P_0}^*} \left(\frac{\partial u_h}{\partial x} dy - \frac{\partial u_h}{\partial y} dx \right) \\
&= - \sum_{i=1}^6 \int_{M_i Q_i M_{i+1}} \left(\frac{\partial u_h}{\partial x} dy - \frac{\partial u_h}{\partial y} dx \right).
\end{aligned} \tag{3.2.17}$$

Since $\frac{\partial u_h}{\partial x}$ and $\frac{\partial u_h}{\partial y}$ are constants on each triangular element K , the

above integral is independent of the position of Q_i . Hence

$$\begin{aligned}
 & a(u_h, \phi_{P_0}) \\
 &= \sum_{i=1}^6 \left[-\frac{\partial u_h(Q_i)}{\partial x} (y_{M_{i+1}} - y_{M_i}) + \frac{\partial u_h(Q_i)}{\partial y} (x_{M_{i+1}} - x_{M_i}) \right] \\
 &= \sum_{i=1}^6 \frac{1}{4S_{Q_i}} \{ [(u_{P_i} - u_{P_0})(y_{P_{i+1}} - y_{P_0}) \\
 &\quad + (u_{P_{i+1}} - u_{P_0})(y_{P_0} - y_{P_i})](y_{P_i} - y_{P_{i+1}}) \\
 &\quad + [(u_{P_i} - u_{P_0})(x_{P_0} - x_{P_{i+1}}) \\
 &\quad + (u_{P_{i+1}} - u_{P_0})(x_{P_i} - x_{P_0})](x_{P_{i+1}} - x_{P_i}) \},
 \end{aligned}$$

where $M_7 = M_1$ and $P_7 = P_1$. For a triangular element K_{Q_i} , write its side lengths $|\overline{P_{i+1}P_0}| = a_i$, $|\overline{P_iP_0}| = b_i$ and $|\overline{P_{i+1}P_i}| = c_i$. Then the difference equation corresponding to P_0 is

$$\begin{aligned}
 a(u_h, \phi_{P_0}) &= \sum_{i=1}^6 \frac{1}{4S_{Q_i}} \left[(u_{P_i} - u_{P_0}) \frac{b_i^2 - c_i^2 - a_i^2}{2} \right. \\
 &\quad \left. + (u_{P_{i+1}} - u_{P_0}) \frac{a_i^2 - b_i^2 - c_i^2}{2} \right] = \int_{K_{P_0}^*} f dx dy.
 \end{aligned} \tag{3.2.18}$$

On the other hand, we note that the integral is independent of the position of the point Q_i . Therefore we can take Q_i as the circumcenter of the triangle. Then it follows from the piecewise linearity of u_h that

$$- \int_{\partial K_{P_0}^*} \frac{\partial u_h}{\partial n} ds = - \sum_{i=1}^6 \int_{Q_i Q_{i+1}} \frac{\partial u_h}{\partial n} ds = - \sum_{i=1}^6 \frac{u_{P_{i+1}} - u_{P_0}}{|\overline{P_{i+1}P_0}|} \cdot \overline{Q_{i+1}Q_i}.$$

So the difference equation related to P_0 becomes

$$a(u_h, \phi_{P_0}) = - \sum_{i=1}^6 \frac{\overline{Q_{i+1}Q_i}}{|\overline{P_{i+1}P_0}|} (u_{P_{i+1}} - u_{P_0}) = \int_{K_{P_0}^*} f dx dy. \tag{3.2.19}$$

(3.2.18) and (3.2.19) are identical. In fact, it can be verified that $\frac{c_i^2 + a_i^2 - b_i^2}{8S_{Q_i}} |\overline{P_iP_0}|$ and $\frac{b_i^2 + c_i^2 - a_i^2}{8S_{Q_i}} |\overline{P_{i+1}P_0}|$ are the distances from the circumcenter of $\triangle P_0 P_i P_{i+1}$ to the sides $\overline{P_iP_0}$ and $\overline{P_{i+1}P_0}$ respectively.

For the triangulation over a non-uniform rectangular mesh as in Fig. 3.2.5, a direct calculation leads to

$$\begin{aligned}
 a(u_h, \phi_{P_0}) &= - \int_{\partial K_{P_{ij}}}^* \frac{\partial u_h}{\partial n} ds \\
 &= - \frac{k_1 + k_2}{2} \left(\frac{\partial u_h}{\partial x} \Big|_{p_{i+1/2,j}} - \frac{\partial u_h}{\partial x} \Big|_{p_{i-1/2,j}} \right) \\
 &\quad - \frac{h_1 + h_2}{2} \left(\frac{\partial u_h}{\partial y} \Big|_{p_{i,j+1/2}} - \frac{\partial u_h}{\partial y} \Big|_{p_{i,j-1/2}} \right) \\
 &= - \frac{k_1 + k_2}{2} \left(\frac{u_{i+1,j} - u_{i,j}}{h_2} + \frac{u_{i+1,j} - u_{i,j}}{h_1} \right) \\
 &\quad - \frac{h_1 + h_2}{2} \left(\frac{u_{i,j+1} - u_{i,j}}{k_2} + \frac{u_{i,j-1} - u_{i,j}}{k_1} \right).
 \end{aligned}$$

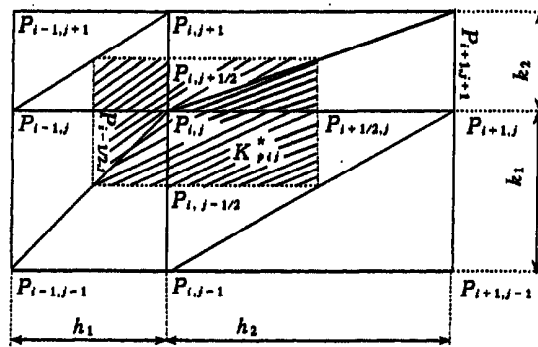


Fig. 3.2.5

So the difference equation corresponding to P_{ij} is

$$\begin{aligned} & \left[\frac{k_1 + k_2}{2} \left(\frac{1}{h_1} + \frac{1}{h_2} \right) + \frac{h_1 + h_2}{2} \left(\frac{1}{k_1} + \frac{1}{k_2} \right) \right] u_{ij} - \\ & \frac{k_1 + k_2}{2h_1} u_{i-1,j} - \frac{k_1 + k_2}{2h_2} u_{i+1,j} - \frac{h_1 + h_2}{2k_1} u_{i,j-1} - \frac{h_1 + h_2}{2k_2} u_{i,j+1} \\ & = \int_{K_{P_{ij}}} f dx dy. \end{aligned} \quad (3.2.20)$$

For the uniform decomposition ($h_1 = h_2 = k_1 = k_2$), (3.2.20) reads

$$4u_{ij} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1} = \int_{K_{P_{ij}}} f dx dy. \quad (3.2.21)$$

This is precisely the five point difference scheme.

Now let us consider an equilateral triangulation as in Fig. 3.2.6. Write $|P_0 P_i| = h$ ($i = 1, \dots, 6$), then $|Q_i Q_{i+1}| = \frac{h}{\sqrt{3}}$, and the difference equation related to P_0 reads

$$\frac{1}{\sqrt{3}} \left(6u_{P_0} - \sum_{i=1}^6 u_{P_i} \right) = \int_{K_{P_0}} f dx dy.$$

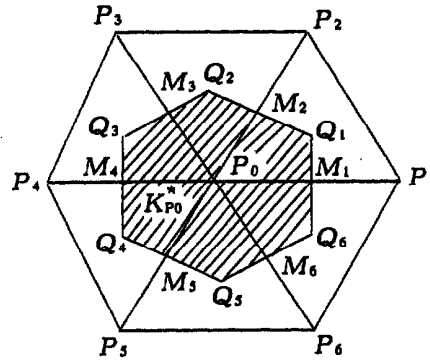


Fig. 3.2.6

3.2.3 a priori estimates

Let us introduce the following discrete zero norm, semi-norm and full-norm:

$$\|u_h\|_{0,h} = \left(\sum_{K \in T_h} |u_h|_{0,h,K}^2 \right)^{1/2}, \quad (3.2.22a)$$

$$|u_h|_{1,h} = \left(\sum_{K \in T_h} |u_h|_{1,h,K}^2 \right)^{1/2}, \quad (3.2.22b)$$

$$\|u_h\|_{1,h} = (\|u_h\|_{0,h}^2 + |u_h|_{1,h}^2)^{1/2}, \quad (3.2.22c)$$

where $K = K_Q = \triangle P_i P_j P_k$ and

$$|u_h|_{0,h,K} = [\tfrac{1}{3}(u_i^2 + u_j^2 + u_k^2)S_Q]^{1/2},$$

$$|u_h|_{1,h,K} = \left\{ \left[\left(\frac{\partial u_h(Q)}{\partial x} \right)^2 + \left(\frac{\partial u_h(Q)}{\partial y} \right)^2 \right] S_Q \right\}^{1/2}.$$

These discrete norms and the continuous norms of the Sobolev spaces have the following relations.

Lemma 3.2.1 For $u_h \in U_h$, $|\cdot|_{1,h}$ and $|\cdot|_1$ are identical; $\|\cdot\|_{0,h}$ and $\|\cdot\|_{1,h}$ are equivalent with $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively, that is, there exist positive constants $c_1, \dots, c_4$ independent of U_h such that

$$c_1 \|u_h\|_{0,h} \leq \|u_h\|_0 \leq c_2 \|u_h\|_{0,h}, \quad \forall u_h \in U_h, \quad (3.2.23a)$$

$$c_3 \|u_h\|_{1,h} \leq \|u_h\|_1 \leq c_4 \|u_h\|_{1,h}, \quad \forall u_h \in U_h. \quad (3.2.23b)$$

Proof The identification of the two norms $|\cdot|_{1,h}$ and $|\cdot|_1$ results from the fact that $\frac{\partial u_h}{\partial x}$ and $\frac{\partial u_h}{\partial y}$ are constants on each element. Since u_h is linear in K we can use the numerical integration formula with second order accuracy to compute that

$$\begin{aligned} & \int_K u_h^2 dx dy \\ &= \frac{1}{3} [u_h^2(M_i) + u_h^2(M_j) + u_h^2(M_k)] S_Q \\ &= \frac{1}{6} (u_i^2 + u_j^2 + u_k^2 + u_i u_j + u_i u_k + u_j u_k) S_Q \\ &= \frac{1}{12} [(u_i^2 + u_j^2 + u_k^2) + (u_i + u_j + u_k)^2] S_Q, \end{aligned}$$

where M_i, M_j and M_k are the midpoints of $\overline{P_j P_k}$, $\overline{P_k P_i}$ and $\overline{P_i P_j}$ respectively (cf. Fig. 3.2.7). This equality gives

$$\frac{1}{4} \|u_h\|_{0,h}^2 \leq \|u_h\|_0^2 \leq \|u_h\|_{0,h}^2.$$

This gives (3.2.23a), and leads to (3.2.23b) thanks to the identification of $|\cdot|_1$ and $|\cdot|_{1,h}$. $\square$

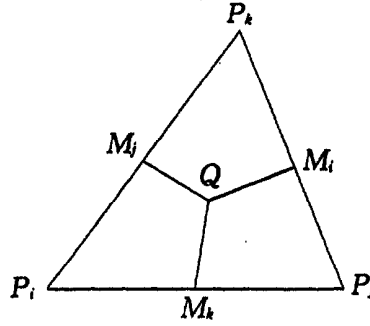


Fig. 3.2.7

Theorem 3.2.1 $a(u_h, \Pi_h^* u_h)$ is positive definite for small enough h , namely, there exist $h_0 > 0$, $\alpha > 0$ such that for $0 < h \leq h_0$

$$a(u_h, \Pi_h^* u_h) \geq \alpha \|u_h\|_1^2, \quad \forall u_h \in U_h. \quad (3.2.24)$$

Proof It follows from (3.2.11) and (3.2.14) that

$$a(u_h, \Pi_h^* u_h) = \sum_{K \in T_h} I_K(u_h, \Pi_h^* u_h), \quad (3.2.25)$$

where

$$\begin{aligned} & I_K(u_h, \Pi_h^* u_h) \\ &= \sum_{P \in \hat{K}} \left[- \int_{\partial K_P^* \cap K} (W_h^{(1)} dy - W_h^{(2)} dx) + \int_{K_P^* \cap K} q u_h dx dy \right] u_h(P), \end{aligned} \quad (3.2.26)$$

where $\dot{K}$ denotes the set of the three vertexes of $K = \triangle P_i P_j P_k$.

First let us prove the positive definiteness of the approximate bilinear form

$$a_h(u_h, \Pi_h^* u_h) = \sum_{K \in T_h} \tilde{I}_K(u_h, \Pi_h^* u_h),$$

where

$$\begin{aligned} & \tilde{I}_K(u_h, \Pi_h^* u_h) \\ = & [W_h^{(1)}(Q)(y_{M_k} - y_{M_j}) + W_h^{(2)}(Q)(x_{M_j} - x_{M_k})]u_h(P_i) \\ & + [W_h^{(1)}(Q)(y_{M_i} - y_{M_k}) + W_h^{(2)}(Q)(x_{M_k} - x_{M_i})]u_h(P_j) \\ & + [W_h^{(1)}(Q)(y_{M_j} - y_{M_i}) + W_h^{(2)}(Q)(x_{M_i} - x_{M_j})]u_h(P_k) \\ & + \sum_{P \in \dot{K}} q(P)u_h(P)S_{K_P^* \cap K} \cdot u_h(P), \end{aligned} \quad (3.2.27)$$

where $S_{K_P^* \cap K}$ denotes the area of $K_P^* \cap K$. It follows from (3.2.6) and (3.2.7) that

$$\begin{aligned} & \tilde{I}_K(u_h, \Pi_h^* u_h) \\ = & \left[a_{11}(Q) \left(\frac{\partial u_h(Q)}{\partial x} \right)^2 + (a_{12}(Q) + a_{21}(Q)) \frac{\partial u_h(Q)}{\partial x} \frac{\partial u_h(Q)}{\partial y} \right. \\ & \left. + a_{22}(Q) \left(\frac{\partial u_h(Q)}{\partial y} \right)^2 \right] S_Q + \sum_{P \in \dot{K}} q(P)u_h^2(P)S_{K_P^* \cap K}. \end{aligned} \quad (3.2.28)$$

By the elliptic condition we have

$$\tilde{I}_K(u_h, \Pi_h^* u_h) \geq r \left[\left(\frac{\partial u_h(Q)}{\partial x} \right)^2 + \left(\frac{\partial u_h(Q)}{\partial y} \right)^2 \right] S_Q.$$

Hence, by Lemma 3.2.1 and the equivalence of the semi-norm and the full norm on H_0^1 , there exists a constant $r' > 0$ such that

$$a_h(u_h, \Pi_h^* u_h) \geq r' \|u_h\|_1^2, \quad \forall u_h \in U_h. \quad (3.2.29)$$

Next we show the positive definiteness of $a(u_h, \Pi_h^* u_h)$. It is easy to see that

$$\begin{aligned}
& I_K(u_h, \Pi_h^* u_h) - \tilde{I}_K(u_h, \Pi_h^* u_h) \\
&= \sum_{P \in \tilde{K}} \left\{ - \int_{\partial K_P^* \cap K} [(W_h^{(1)} - W_h^{(1)}(Q)) dy \right. \\
&\quad \left. - (W_h^{(2)} - W_h^{(2)}(Q)) dx] \right. \\
&\quad \left. + \int_{K_P^* \cap K} (qu_h - q(P)u_h(P)) dx dy \right\} u_h(P) \\
&= \left\{ \sum_{l=i,j,k} \int_{M_l Q} [(W_h^{(1)} - W_h^{(1)}(Q)) dy \right. \\
&\quad \left. - (W_h^{(2)} - W_h^{(2)}(Q)) dx] (u_{l+2} - u_{l+1}) \right. \\
&\quad \left. + \sum_{P \in \tilde{K}} \int_{K_P^* \cap K} (qu_h - q(P)u_h(P)) dx dy \right\} \cdot u_h(P),
\end{aligned} \tag{3.2.30}$$

where we set $u_{i+1} = u_j$, $u_{j+1} = u_k$, $u_{k+1} = u_i$, and $u_l = u_h(P_l)$. Since $\frac{\partial u_h}{\partial x}$ and $\frac{\partial u_h}{\partial y}$ are constants in K we have

$$\begin{aligned}
& |W_h^{(i)} - W_h^{(i)}(Q)| \\
&= \left| (a_{i1} - a_{i1}(Q)) \frac{\partial u_h}{\partial x} + (a_{i2} - a_{i2}(Q)) \frac{\partial u_h}{\partial y} \right| \\
&\leq Ch \left(\left| \frac{\partial u_h}{\partial y} \right| + \left| \frac{\partial u_h}{\partial x} \right| \right), \quad i = 1, 2.
\end{aligned} \tag{3.2.31}$$

Noticing the linearity of u_h in K and employing the Taylor expansion we have

$$\begin{aligned}
& |u_{l+2} - u_{l+1}| \\
&= \left| \frac{\partial u_h}{\partial x} (x_{P_{l+2}} - x_{P_{l+1}}) + \frac{\partial u_h}{\partial y} (y_{P_{l+2}} - y_{P_{l+1}}) \right| \\
&\leq h \left(\left| \frac{\partial u_h}{\partial x} \right| + \left| \frac{\partial u_h}{\partial y} \right| \right), \quad l = i, j, k.
\end{aligned} \tag{3.2.32}$$

By (3.2.31), (3.2.32) and the quasi-uniformity of the decomposition we have

$$\begin{aligned}
& \left| \int_{M_l Q} [(W_h^{(1)} - W_h^{(1)}(Q)) dy \right. \\
& \quad \left. - (W_h^{(2)} - W_h^{(2)}(Q)) dx] (u_{l+2} - u_{l+1}) \right| \\
& \leq Ch^3 \left(\left| \frac{\partial u_h}{\partial x} \right| + \left| \frac{\partial u_h}{\partial y} \right| \right)^2 \\
& \leq Ch \left[\left(\frac{\partial u_h}{\partial x} \right)^2 + \left(\frac{\partial u_h}{\partial y} \right)^2 \right] S_Q.
\end{aligned} \tag{3.2.33}$$

On $K_{P_l}^* \cap K$

$$u_h = u_h(P_l) + \frac{\partial u_h}{\partial x}(x - x_{P_l}) + \frac{\partial u_h}{\partial y}(y - y_{P_l}),$$

$$|qu_h - q(P)u_h(P)| \leq |(q - q(P))u_h| + |q(P)(u_h - u_h(P))|,$$

$$|(q - q(P_l))u_h| \leq Ch \left(|u_h(P_l)| + h \left| \frac{\partial u_h}{\partial x} \right| + h \left| \frac{\partial u_h}{\partial y} \right| \right),$$

$$|q(P_l)(u_h - u_h(P_l))| \leq Ch \left(\left| \frac{\partial u_h}{\partial x} \right| + \left| \frac{\partial u_h}{\partial y} \right| \right), \quad l = i, j, k.$$

So

$$\begin{aligned}
& \left| \int_{K_{P_l}^* \cap K} (qu_h - q(P_l)u_h(P_l)) dx dy \cdot u_h(P_l) \right| \\
& \leq Ch \left[(u_l)^2 + \left(\frac{\partial u_h}{\partial x} \right)^2 + \left(\frac{\partial u_h}{\partial y} \right)^2 \right] S_Q.
\end{aligned} \tag{3.2.34}$$

It follows from (3.2.30), (3.2.33), (3.2.34) and Lemma 3.2.1 that

$$\begin{aligned}
& |a(u_h, \Pi_h^* u_h) - a_h(u_h, \Pi_h^* u_h)| \\
& = \left| \sum_{k \in T_h} [I_K(u_h, \Pi_h^* u_h) - \tilde{I}_K(u_h, \Pi_h^* u_h)] \right| \\
& \leq Ch \|u_h\|_1^2.
\end{aligned} \tag{3.2.35}$$

Combining (3.2.29) and (3.2.35) leads to (3.2.24). $\square$

From Theorem 3.2.1 it is easy to deduce the existence and uniqueness of the solution to the generalized difference scheme (3.2.13).

3.2.4 Error estimates

Theorem 3.2.2 *Let u be the generalized solution to (3.1.1) and u_h the solution to the generalized difference scheme (3.2.13). If $u \in H^2(\Omega)$, then the following error estimate holds:*

$$\|u - u_h\|_1 \leq Ch\|u\|_2. \quad (3.2.36)$$

Proof It is obvious that

$$a(u - u_h, \psi_{P_0}) = 0, \quad \forall P_0 \in \hat{\Omega}_h, \quad (3.2.37)$$

which together with Theorem 3.2.1 yields

$$\begin{aligned} & \|u_h - \Pi_h u\|_1^2 \\ & \leq \frac{1}{\alpha} a(u_h - \Pi_h u_h, \Pi_h^*(u_h - \Pi_h u)) \\ & = \frac{1}{\alpha} a(u - \Pi_h u_h, \Pi_h^*(u_h - \Pi_h u)). \end{aligned}$$

So

$$\|u_h - \Pi_h u\|_1 \leq \frac{1}{\alpha} \sup_{\bar{u}_h \in U_h} \frac{|a(u - \Pi_h u_h, \Pi_h^* \bar{u}_h)|}{\|\bar{u}_h\|_1}.$$

By (3.2.25) and (3.2.26) we have

$$a(u - \Pi_h u_h, \Pi_h^* \bar{u}_h) = \sum_{K \in T_h} I_K(u - \Pi_h u_h, \Pi_h^* \bar{u}_h), \quad (3.2.39)$$

$$\begin{aligned} & I_K(u - \Pi_h u_h, \Pi_h^* \bar{u}_h) \\ & = \sum_{l=i,j,k} \left\{ \int_{\bar{M}_l \bar{Q}} [\bar{W}_h^{(1)} \cos \langle n_l, x \rangle + \bar{W}_h^{(2)} \cos \langle n_l, y \rangle] ds \right. \\ & \quad \left. \cdot (\bar{u}_{l+2} - \bar{u}_{l+1}) + \int_{K_{P_l}^* \cap K} q(u - \Pi_h u) dx dy \cdot \bar{u}_h(P_l) \right\}, \end{aligned} \quad (3.2.40)$$

where $\bar{W}_h^{(i)} = a_{i1} \frac{\partial(u - \Pi_h u)}{\partial x} + a_{i2} \frac{\partial(u - \Pi_h u)}{\partial y}$ ($i = 1, 2$) and n_l is the unit outer normal vector of $K_{P_l}^* \cap K$ along $\bar{M}_l \bar{Q}$ ($l = i, j, k$). It follows from (3.2.32) that

$$|\bar{u}_{l+2} - \bar{u}_{l+1}| \leq h \left(\left| \frac{\partial \bar{u}_h}{\partial x} \right| + \left| \frac{\partial \bar{u}_h}{\partial y} \right| \right) \leq C |\bar{u}_h|_{1,h,K}. \quad (3.2.41)$$

On the other hand,

$$\begin{aligned}
& \left| \int_{M_l Q} [\bar{W}_h^{(1)} \cos \langle n_l, x \rangle + \bar{W}_h^{(2)} \cos \langle n_l, y \rangle] ds \right| \\
& \leq C \int_{M_l Q} \left(\left| \frac{\partial(u - \Pi_h u)}{\partial x} \right| + \left| \frac{\partial(u - \Pi_h u)}{\partial y} \right| \right) ds \\
& \leq Ch^{1/2} \left\{ \int_{M_l Q} \left(\left| \frac{\partial(u - \Pi_h u)}{\partial x} \right|^2 + \left| \frac{\partial(u - \Pi_h u)}{\partial y} \right|^2 \right) ds \right\}^{1/2}.
\end{aligned} \tag{3.2.42}$$

Set $\phi_1 = \frac{\partial(u - \Pi_h u)}{\partial x}$ and $\phi_2 = \frac{\partial(u - \Pi_h u)}{\partial y}$. The mapping $(x, y) \rightarrow (\lambda_j, \lambda_k)$ maps the element K onto the reference element $\hat{K}$; the function ϕ_m on K into the function $\hat{\phi}_m(\lambda_j, \lambda_k) = \phi_m(x, y)$ ($m = 1, 2$); and the points M_l, P_l, Q into $\hat{M}_l, \hat{P}_l, \hat{Q}$ ($l = i, j, k$) respectively. It is obvious that

$$\int_{M_l Q} |\phi_m|^2 ds \leq h \int_{\hat{M}_l \hat{Q}} |\hat{\phi}_m|^2 d\hat{s}, \quad m = 1, 2.$$

By the trace theorem on $\hat{K}_{P_{l+1}}^* \cap \hat{K}$ we have a constant $C > 0$ independent of K such that

$$\int_{\hat{M}_l \hat{Q}} |\hat{\phi}_m|^2 d\hat{s} \leq C \|\hat{\phi}_m\|_{1, \hat{K}}^2, \quad m = 1, 2.$$

Using Theorem 1.1.12 we have

$$|\hat{\phi}_m|_{0, \hat{K}} \leq Ch^{-1} |\phi_m|_{0, K}, \quad |\hat{\phi}_m|_{1, \hat{K}} \leq C |\phi_m|_{1, K}, \quad m = 1, 2.$$

Hence

$$\begin{aligned}
& \int_{M_l Q} |\phi_m|^2 ds \leq Ch(h^{-1} |\phi_m|_{0, K} + |\phi_m|_{1, K})^2 \\
& \leq Ch(h^{-1} |u - \Pi_h u|_{1, K} + |u - \Pi_h u|_{2, K})^2 \leq Ch |u|_{2, K}^2, \quad m = 1, 2.
\end{aligned} \tag{3.2.43}$$

It follows from (3.2.41)-(3.2.43) that

$$\begin{aligned}
& \left| \sum_{l=i,j,k} \int_{M_l Q} [\bar{W}_h^{(1)} \cos \langle n_l, x \rangle + \bar{W}_h^{(2)} \cos \langle n_l, y \rangle] ds (\bar{u}_{l+2} - \bar{u}_{l+1}) \right| \\
& \leq Ch |u|_{2, K} |\bar{u}_h|_{1, K}.
\end{aligned} \tag{3.2.44}$$

It is easy to deduce that

$$\begin{aligned}
 & \left| \sum_{l=i,j,k} \int_{K_{P_l}^* \cap K} q(u - \Pi_h u) dx dy \cdot \bar{u}_h(P_l) \right| \\
 & \leq C \sum_{l=i,j,k} \int_{K_{P_l}^* \cap K} |u - \Pi_h u| dx dy \cdot |\bar{u}_h(P_l)| \\
 & \leq Ch^2 |u|_{2,K} |\bar{u}_h|_{0,K}.
 \end{aligned} \tag{3.2.45}$$

It follows from (3.2.39), (3.2.40), (3.2.44) and (3.2.45) that

$$|a(u - \Pi_h u, \Pi_h^* \bar{u}_h)| \leq Ch |u|_2 \|\bar{u}_h\|_1. \tag{3.2.46}$$

A combination of (3.2.38) and (3.2.46) leads to

$$\|u_h - \Pi_h u\|_1 \leq Ch |u|_2.$$

This together with (3.2.8) implies (3.2.36) and completes the proof. $\square$

3.3 Generalized Difference Methods on Quadrilateral Meshes

3.3.1 Trial and test function spaces

Suppose Ω is a polygonal region, of which the boundary $\partial\Omega$ is a simple closed fold line. Divide $\bar{\Omega}$ into a sum of finite number of strictly convex quadrilaterals such that different quadrilaterals have no common interior point, that a vertex of any quadrilateral does not lie on an interior of a side of any other quadrilateral and that any vertex of the boundary is a vertex of some quadrilateral. Each quadrilateral is called an element and denoted by K . All the elements constitute a quadrilateral decomposition of $\bar{\Omega}$, denoted by T_h where h is the largest diameter of all the quadrilaterals. The vertexes of the quadrilaterals are called the nodes of the decomposition. Two nodes are adjacent if they are the two endpoints of a certain side of an element. Two elements are adjacent if they share a common side.

Next we construct the dual decomposition related to T_h . As in Fig. 3.3.1, let P_0 be a node of the decomposition T_h , P_i ($i = 1, \dots, 4$) the adjacent nodes of P_0 , M_i the midpoint of $\overline{P_0 P_i}$, and $P_{i,i+1}$ (convention: $P_{45} = P_{41} = P_{14}$) the vertex facing P_0 , in the quadrilateral with sides $\overline{P_0 P_i}$ and $\overline{P_0 P_{i+1}}$. Take any point Q_i in the quadrilateral $P_0 P_i P_{i,i+1} P_{i+1}$ ($P_5 = P_1$), connect successively $M_1, Q_1, M_2, Q_2, \dots, M_4, Q_4, M_1$ to form a polygonal region $K_{P_0}^*$, called a dual element. All the dual elements constitute a new decomposition, the dual decomposition, T_h^* of $\bar{\Omega}$. Q_i and M_i are called the nodes of the dual decomposition. The most important dual decomposition takes Q_i as the joint of the two lines connecting the midpoints of the opposite sides of the quadrilateral element. This is called the central dual decomposition.

As in §3.2, let $\bar{\Omega}_h$ be the node set of the decomposition T_h and $\dot{\Omega}_h = \bar{\Omega}_h \setminus \partial\Omega$ the set of all interior nodes. Ω_h^* denotes the node set of the dual decomposition T_h^* . For $Q \in \Omega_h^*$, we denote by K_Q the quadrilateral element containing Q . S_Q (or S_{K_Q}) and $S_{P_0}^*$ stand for the areas of the quadrilateral element K_Q and the dual element $K_{P_0}^*$ respectively. We shall always assume that T_h and T_h^* are quasi-uniform such that there exist constants $c_1, c_2, c_3 > 0$ independent of h such that

$$c_1 h^2 \leq S_Q \leq h^2, \quad Q \in \Omega_h^*, \quad (3.3.1a)$$

$$c_2 h^2 \leq S_{P_0}^* \leq c_3 h^2, \quad P_0 \in \bar{\Omega}_h. \quad (3.3.1b)$$

We point out that for the central dual decomposition, (3.3.1b) can be deduced from (3.3.1a).

The trial function space U_h is chosen as the isoparametric element space of the bilinear functions on a quadrilateral decomposition T_h (cf. [B-17]). Its construction is as follows. Take the unit square $\hat{K} = \{(\xi, \eta) : 0 \leq \xi, \eta \leq 1\}$ of the (ξ, η) plane as a reference element. For a quadrilateral element K_Q , suppose its vertexes are $P_i(x_i, y_i)$ ($i = 1, \dots, 4$). Then there exists a unique invertible bilinear mapping

$$F_{K_Q} : \begin{cases} x = x_1 + a_1 \xi + a_2 \eta + a_3 \xi \eta, \\ y = y_1 + b_1 \xi + b_2 \eta + b_3 \xi \eta, \end{cases} \quad (3.3.2)$$

where

$$\begin{aligned} a_1 &= x_2 - x_1, \quad a_2 = x_3 - x_1, \quad a_3 = x_4 - x_3 - x_2 + x_1, \\ b_1 &= y_2 - y_1, \quad b_2 = y_3 - y_1, \quad b_3 = y_4 - y_3 - y_2 + y_1, \end{aligned}$$

which maps $\hat{K}$ onto K_Q (cf. Fig. 3.3.2).

For any $u_h \in U_h$ we have on K_Q that

$$\begin{aligned} u_h &= P_{\hat{K}}(\xi, \eta) \\ &= u_1(1 - \xi)(1 - \eta) + u_2\xi(1 - \eta) + u_3(1 - \xi)\eta + u_4\xi\eta \\ &= u_1 + (u_2 - u_1)\xi + (u_3 - u_1)\eta + (u_4 - u_3 - u_2 + u_1)\xi\eta. \end{aligned} \quad (3.3.3)$$

So we have

$$U_h = \{u_h \in C(\bar{\Omega}) : u_h|_{K_Q} = P_{\hat{K}} \circ F_{K_Q}^{-1}, u_h|_{\partial\Omega} = 0, P_{\hat{K}} \in \mathcal{P}_{11}\},$$

where $\mathcal{P}_{11}$ is the family of bilinear functions.

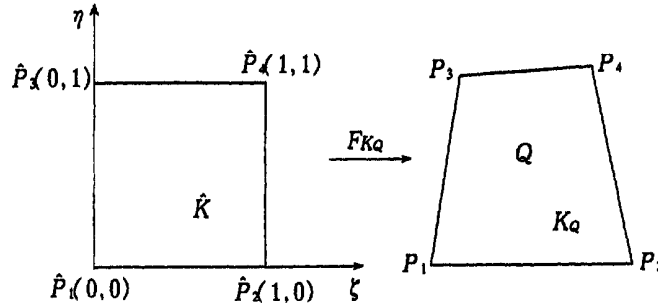


Fig. 3.3.2

The finite element obtained through the transformation F_{K_Q} is called the quadrilateral isoparametric element with four nodes. If $u \in H^2(\Omega)$ and $\Pi_h u$ is its interpolation projection onto U_h , then

$$|u - \Pi_h u|_m \leq Ch^{2-m}|u|_2, \quad m = 0, 1. \quad (3.3.4)$$

The test space V_h is chosen as the piecewise constant function space related to the dual decomposition, with the following basis functions: For each $P_0 \in \dot{\Omega}_h$,

$$\psi_{P_0}(P) = \begin{cases} 1, & P \in K_{P_0}^*, \\ 0, & P \notin K_{P_0}^*. \end{cases}$$

So ψ_{P_0} is the characteristic function of the dual element $K_{P_0}^*$. For any $v_h \in V_h$ we have

$$v_h = \sum_{P_0 \in \dot{\Omega}_h} v_h(P_0) \psi_{P_0}. \quad (3.3.5)$$

3.3.2 Generalized difference equation

The generalized difference equation corresponding to the above trial and test function spaces is: Find $u_h \in U_h$ such that

$$a(u_h, \psi_{P_0}) = (f, \psi_{P_0}), \quad \forall P_0 \in \dot{\Omega}_h, \quad (3.3.6)$$

where

$$a(u_h, \psi_{P_0}) = - \int_{\partial K_{P_0}^*} (W_h^{(1)} dy - W_h^{(2)} dx) + \int_{K_{P_0}^*} q u_h dx dy, \quad (3.3.7)$$

where $\partial K_{P_0}^*$ is the boundary of $K_{P_0}^*$, possessing a counterclockwise direction, and

$$W_h^{(i)} = a_{i1} \frac{\partial u_h}{\partial x} + a_{i2} \frac{\partial u_h}{\partial y}, \quad i = 1, 2.$$

Using different numerical integration formulas to compute the integrals of the right-hand side of (3.3.7) leads to different approximations $a_h(u_h, \psi_{P_0})$ of $a(u_h, \psi_{P_0})$ and results in different difference equations:

$$a_h(u_h, \psi_{P_0}) = (f, \psi_{P_0}), \quad P_0 \in \dot{\Omega}_h. \quad (3.3.8)$$

Let the dual decomposition be as in Fig. 3.3.1. Then the first integral of the right-hand side of (3.3.7) can be divided into a sum of line integrals along $\overline{M_1 Q_1}$, $\overline{Q_1 M_2}$, $\dots$, $\overline{Q_4 M_1}$. For each line integral on

$\overline{M_j Q_j}$ (or $\overline{Q_j M_{j+1}}$), if we use its value at M_i , or Q_i , or their average respectively to replace the integral function, then we obtain three approximations $a_h^k(u_h, \psi_{P_0})$ ($k = 1, 2, 3$) of $a(u_h, \psi_{P_0})$, ending up with three difference equations. One can also divide the line integral in (3.3.7) into a sum of integrals on $\overline{M_1 Q_1 M_2}$, $\overline{M_2 Q_2 M_3}$, $\overline{M_3 Q_3 M_4}$ and $\overline{M_4 Q_4 M_1}$, and approximate $W_h^{(i)}$ by $W_h^{(i)}(Q_j)$ ($i = 1, 2, j = 1, 2, 3, 4$), then we have the following difference scheme (cf. [B-32]):

$$\begin{aligned}
& -W_h^{(1)}(Q_1)(y_{M_2} - y_{M_1}) - W_h^{(1)}(Q_2)(y_{M_3} - y_{M_2}) \\
& -W_h^{(1)}(Q_3)(y_{M_4} - y_{M_3}) - W_h^{(1)}(Q_4)(y_{M_1} - y_{M_4}) \\
& +W_h^{(2)}(Q_1)(x_{M_2} - x_{M_1}) + W_h^{(2)}(Q_2)(x_{M_3} - x_{M_2}) \\
& +W_h^{(2)}(Q_3)(x_{M_4} - x_{M_3}) + W_h^{(2)}(Q_4)(x_{M_1} - x_{M_4}) \\
& +q(P_0)u_h(P_0)S_{P_0}^* \\
& = f(P_0)S_{P_0}^*, \quad \forall P_0 \in \dot{\Omega}_h.
\end{aligned} \tag{3.3.9}$$

Define the following difference operators

$$\begin{aligned}
(\nabla_1 \phi)_{P_0} &= [\phi(Q_1)(y_{M_2} - y_{M_1}) + \phi(Q_2)(y_{M_3} - y_{M_2}) \\
& + \phi(Q_3)(y_{M_4} - y_{M_3}) + \phi(Q_4)(y_{M_1} - y_{M_4})]/S_{P_0}^*, \\
(\nabla_2 \phi)_{P_0} &= [\phi(Q_1)(x_{M_2} - x_{M_1}) + \phi(Q_2)(x_{M_3} - x_{M_2}) \\
& + \phi(Q_3)(x_{M_4} - x_{M_3}) + \phi(Q_4)(x_{M_1} - x_{M_4})]/S_{P_0}^*.
\end{aligned}$$

Then (3.3.9) can be rewritten as a conservation form:

$$-\sum_{i=1}^2 \left(\nabla_i \left(\sum_{j=1}^2 a_{ij} \frac{\partial u_h}{\partial x_j} \right) \right)_{P_0} + q(P_0)u_h(P_0) = f(P_0), \quad \forall P_0 \in \dot{\Omega}_h, \tag{3.3.9}'$$

where $\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial x_2} = \frac{\partial}{\partial y}$.

If T_h is a rectangular decomposition and the sides of the rectangles are parallel to the coordinate axes, then the dual decomposition is also a rectangular decomposition (cf. Fig. 3.3.3), and the above

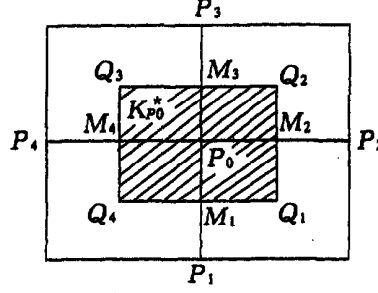


Fig. 3.3.3

mentioned three kinds of numerical integrations lead to the following three difference schemes respectively.

Scheme I:

$$\begin{aligned}
 & -[W_h^{(1)}(M_2^-) - W_h^{(1)}(M_4^-)](y_{P_0} - y_{M_1}) \\
 & -[W_h^{(1)}(M_2^+) - W_h^{(1)}(M_4^+)](y_{M_3} - y_{P_0}) \\
 & +[W_h^{(2)}(M_1^-) - W_h^{(2)}(M_3^-)](x_{P_0} - x_{M_4}) \\
 & +[W_h^{(2)}(M_1^+) - W_h^{(2)}(M_3^+)](x_{M_2} - x_{P_0}) \\
 & +q(P_0)u_h(P_0)S_{P_0}^* = f(\bar{P})S_{P_0}^*, \quad \forall P_0 \in \hat{\Omega}_h,
 \end{aligned}$$

where $W_h^{(1)}(M_i^-)$ and $W_h^{(1)}(M_i^+)$ ($i = 2, 4$) denote the single-sided limits of $W_h^{(1)}$ at M_i from left and right sides respectively along $\overline{Q_i Q_{i+1}}$ ($i = 1, 3$); $W_h^{(2)}(M_i^-)$ and $W_h^{(2)}(M_i^+)$ ($i = 1, 3$) stand for the single-sided limits of $W_h^{(2)}$ at M_i from down side and upside respectively along $\overline{Q_4 Q_1}$ and $\overline{Q_2 Q_3}$; $\bar{P}$ can be viewed as an averaging center of the rectangle K_P^* ; and the meanings of the other notations are self-evident. In particular, Scheme I becomes the well-known five-point difference scheme when A is the Laplacian operator.

Scheme II:

$$\begin{aligned}
& -[W_h^{(1)}(Q_1) - W_h^{(1)}(Q_4)](y_{P_0} - y_{M_1}) \\
& -[W_h^{(1)}(Q_2) - W_h^{(1)}(Q_3)](y_{M_3} - y_{P_0}) \\
& +[W_h^{(2)}(Q_4) - W_h^{(2)}(Q_3)](x_{P_0} - x_{M_4}) \\
& +[W_h^{(2)}(Q_1) - W_h^{(2)}(Q_2)](x_{M_2} - x_{P_0}) \\
& +q(P_0)u_h(P_0)S_{P_0}^* = f(\bar{P})S_{P_0}^*, \quad \forall P_0 \in \dot{\Omega}_h,
\end{aligned}$$

Scheme III:

$$\begin{aligned}
& \frac{1}{2}[W_h^{(1)}(Q_1) + W_h^{(1)}(M_2^-) - W_h^{(1)}(Q_4) + W_h^{(1)}(M_4^-)](y_{P_0} - y_{M_1}) \\
& -\frac{1}{2}[W_h^{(1)}(Q_2) + W_h^{(1)}(M_2^+) - W_h^{(1)}(Q_3) - W_h^{(1)}(M_4^+)](y_{M_3} - y_{P_0}) \\
& +\frac{1}{2}[W_h^{(2)}(Q_4) + W_h^{(2)}(M_1^-) - W_h^{(2)}(Q_3) - W_h^{(2)}(M_3^-)](x_{P_0} - x_{M_4}) \\
& +\frac{1}{2}[W_h^{(2)}(Q_1) + W_h^{(2)}(M_1^+) - W_h^{(2)}(Q_2) - W_h^{(2)}(M_3^+)](x_{M_2} - x_{P_0}) \\
& +\frac{1}{4}\sum_{i=1}^4 q(Q_i)u_h(Q_i)S_{P_0}^* = f(\bar{P})S_{P_0}^*, \quad \forall P_0 \in \dot{\Omega}_h.
\end{aligned}$$

3.3.3 Convergence order estimates

Suppose that T_h and T_h^* are a quasi-uniform arbitrary quadrilateral grid and its central dual grid respectively, and that u and u_h are the solutions to the Poisson equation and the corresponding generalized difference scheme (3.3.6) respectively. Then under certain geometrical restrictions on the quadrilateral grid, there holds the following error estimate (see [B-62])

$$\|u - u_h\|_1 \leq Ch|u|_2. \quad (3.3.10)$$

In the case of rectangular grid and under stronger assumptions on the smoothness of the solutions, a higher order convergence estimate, namely a superconvergence result, in a discrete norm as follows can be obtained. (See [A-62] for details.)

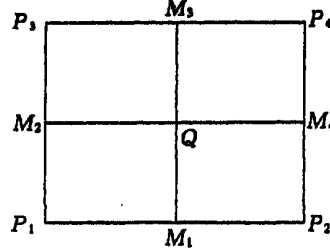


Fig. 3.3.4

Theorem 3.3.1 Let T_h and T_h^* be a quasi-uniform rectangular grid and its central dual grid respectively, and let u and u_h be the solutions to (3.1.1) and the difference scheme (3.3.6) respectively. If $u \in C^3(\bar{\Omega})$, then the following error estimate holds:

$$\|u - u_h\|_{1,h} \leq Ch^2 M_{23}, \quad (3.3.11)$$

where $M_{23} = \max\{|D^2 u|_{\max}, |D^3 u|_{\max}\}$, and the discrete norm is defined by (cf. Fig. 3.3.4)

$$\begin{aligned} \|u\|_{1,h} &= (|u|_{0,h}^2 + |u|_{1,h}^2)^{1/2}, \\ |u|_{m,h} &= \left(\sum_{K \in T_h} |u|_{m,h,K}^2 \right)^{1/2}, \quad m = 0, 1, \\ |u|_{0,h,K} &= \left\{ \frac{1}{4} [u^2(P_1) + u^2(P_2) + u^2(P_3) + u^2(P_4)] S_Q \right\}^{1/2}, \\ |u|_{1,h,K} &= \left\{ \left[\left(\frac{\partial u(M_1)}{\partial x} \right)^2 + \left(\frac{\partial u(M_3)}{\partial x} \right)^2 \right. \right. \\ &\quad \left. \left. + \left(\frac{\partial u(M_2)}{\partial y} \right)^2 + \left(\frac{\partial u(M_4)}{\partial y} \right)^2 \right] S_Q \right\}^{1/2}. \end{aligned}$$

For the above Schemes I and II, if the difference of the squares of any two successive step-lengths on x and y directions is $O(h^{2+d})$, we have the following error estimate:

$$\|u - u_h\|_{1,h} \leq Ch^{1+d} \max_{1 \leq l \leq 4} |D^l u|_{\max}, \quad 0 \leq d \leq 1. \quad (3.3.12)$$

For Scheme III, the following error estimate holds for the quasi-uniform rectangular mesh:

$$\|u - u_h\|_{1,h} \leq Ch^2 \max_{1 \leq l \leq 4} |D^l u|_{\max}. \quad (3.3.13)$$

For Scheme (3.3.9) on arbitrary quadrilateral meshes, the convergence order is $O(h)$ under suitable assumptions on the decomposition. (cf. [B-32] and [C-7]).

3.4 Quadratic Element Difference Schemes

The following two sections will be devoted to the generalized difference methods based on higher order elements. For simplicity, we take the boundary value problem of the Poisson equation as an example to illustrate the idea.

Let Ω be a planar polygonal region with boundary $\partial\Omega$ and $f \in L^2(\Omega)$. Consider the first boundary value problem of the Poisson equation:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \quad (3.4.1a)$$

$$(3.4.1b)$$

The corresponding variational problem is: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \quad (3.4.2)$$

where

$$a(u, v) = \int_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx dy. \quad (3.4.3)$$

3.4.1 Trial and test function spaces

As in §3.2, let T_h be a quasi-uniform triangulation of $\bar{\Omega}$. T_h consists of finite number of triangular elements K_Q , Q being the barycenter of the triangle. The vertexes of the triangles and the midpoints of the sides are taken as the nodes. $\bar{\Omega}_h$ denotes the set of the vertexes of all the triangular elements, $\bar{M}_h$ the set of the midpoints of the sides

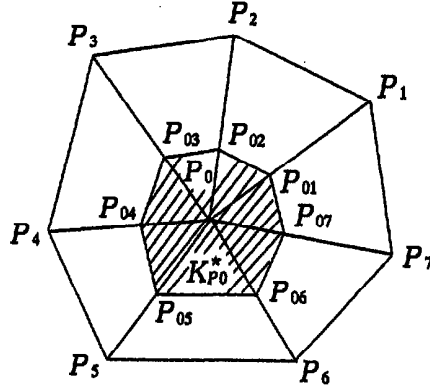


Fig. 3.4.1

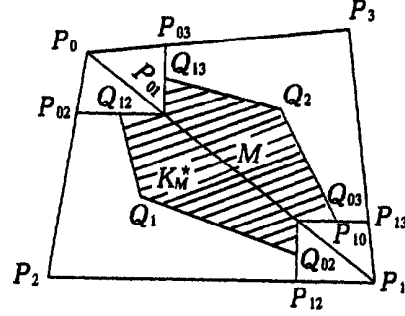


Fig. 3.4.2

of all the elements, Ω_h^* the set of the barycenters of all the elements, and $\bar{\Omega}_h = \bar{\Omega}_h \setminus \partial\Omega$, $M_h = \bar{M}_h \setminus \partial\Omega$.

The dual decomposition of T_h is denoted by T_h^* , consisting of the polygons $K_{P_0}^*$ surrounding the node $P_0 \in \bar{\Omega}_h$ and K_M^* surrounding $M \in \bar{M}_h$. These small polygons are called dual elements. Their detailed construction is as follows.

1) Construction of $K_{P_0}^*$. Suppose that $P_0 \in \bar{\Omega}_h$, that P_i ($i = 1, 2, \dots, 7$) are its adjacent vertexes, and that P_{0i} is a point on $\overline{P_0P_i}$ such that $\overline{P_0P_{0i}} = \frac{1}{3}\overline{P_0P_i}$. Connect successively P_{0i} ($i = 1, 2, \dots, 7$) to obtain a polygon $K_{P_0}^*$ surrounding P_0 . (See Fig. 3.4.1.)

2) Construction of K_M^* . Let $M \in \bar{M}_h$ be a midpoint of a common side of two adjacent triangular elements $K_{Q_1} = \triangle P_0P_1P_2$ and $K_{Q_2} = \triangle P_0P_1P_3$. Denote by $Q_{12}, Q_{13}, Q_{02}, Q_{03}$ the midpoints of $\overline{P_{01}P_{02}}, \overline{P_{01}P_{03}}, \overline{P_{10}P_{12}}$ and $\overline{P_{10}P_{13}}$ respectively. A polygon K_M^* surrounding M is obtained by connecting successively $P_{10}, Q_{03}, Q_2, Q_{13}, P_{01}, Q_{12}, Q_1, Q_{02}, P_{10}$ (see Fig. 3.4.2).

The trial space U_h is chosen as the Lagrangian quadratic element space related to the triangulation T_h . For each $P_0 \in \bar{\Omega}_h$ and $M_0 \in \bar{M}_h$, the corresponding basis functions are the piecewise quadratic polynomials satisfying the following interpolation condi-

tions respectively:

$$\phi_{P_0}(P) = \begin{cases} 1, & P = P_0, \\ 0, & P \in \bar{\Omega}_h \cup \bar{M}_h \setminus \{P_0\}, \end{cases} \quad (3.4.4a)$$

$$\phi_{M_0}(P) = \begin{cases} 1, & P = M_0, \\ 0, & P \in \bar{\Omega}_h \cup \bar{M}_h \setminus \{M_0\}. \end{cases} \quad (3.4.4b)$$

So $U_h = \text{span}\{\phi_{P_0}, \phi_{M_0}; P_0 \in \bar{\Omega}_h, M \in \bar{M}_h\}$.

The test function space V_h is taken as the piecewise constant function space related to the dual decomposition T_h^* . For each $P_0 \in \bar{\Omega}_h$ and $M_0 \in \bar{M}_h$, the corresponding basis functions are the characteristic functions of $K_{P_0}^*$ and $K_{M_0}^*$ respectively:

$$\psi_{P_0}(P) = \begin{cases} 1, & P \in K_{P_0}^*, \\ 0, & P \notin K_{P_0}^*, \end{cases} \quad (3.4.5a)$$

$$\psi_{M_0}(P) = \begin{cases} 1, & P \in K_{M_0}^*, \\ 0, & P \notin K_{M_0}^*. \end{cases} \quad (3.4.5b)$$

Hence $V_h = \text{span}\{\psi_{P_0}, \psi_{M_0}; P_0 \in \bar{\Omega}_h, M \in \bar{M}_h\}$.

3.4.2 Generalized difference equation

The quadratic element difference scheme corresponding to U_h and V_h constructed above is: Find $u_h \in U_h$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad (3.4.6)$$

or equivalently

$$\begin{cases} a(u_h, \psi_{P_0}) = (f, \psi_{P_0}), & P_0 \in \bar{\Omega}_h, \\ a(u_h, \psi_M) = (f, \psi_M), & M \in \bar{M}_h, \end{cases} \quad \begin{matrix} (3.4.6a)' \\ (3.4.6b)' \end{matrix}$$

where

$$a(u_h, v_h) = \sum_{P_0 \in \bar{\Omega}_h} v_h(P_0) a(u_h, \psi_{P_0}) + \sum_{M \in \bar{M}_h} v_h(M) a(u_h, \psi_M), \quad (3.4.7a)$$

$$a(u_h, \psi_{P_0}) = - \int_{\partial K_{P_0}^*} \frac{\partial u_h}{\partial x} dy - \frac{\partial u_h}{\partial y} dx, \quad (3.4.7b)$$

$$a(u_h, \psi_M) = - \int_{\partial K_M^*} \frac{\partial u_h}{\partial x} dy - \frac{\partial u_h}{\partial y} dx. \quad (3.4.7c)$$

Note (cf. Figs. 3.4.1 and 3.4.2) that $\partial K_{P_0}^* = \overline{P_{01}P_{02}} \cup \overline{P_{02}P_{03}} \cup \dots \cup \overline{P_{07}P_{01}}$ and $\partial K_M^* = \overline{Q_1Q_{02}} \cup \overline{Q_{02}P_{10}} \cup \dots \cup \overline{Q_{12}Q_1}$. The right-hand side integrals of (3.4.7b) and (3.4.7c) can be divided into a sum of the easy-to-compute integrals on these segments, resulting in a linear algebraic system with unknowns $u_h(P_0)$ ($P_0 \in \Omega_h$) and $u_h(M)$ ($M \in \dot{M}_h$).

There are two approaches to form the generalized difference equation: Directly compute the equation for each node, or first compute the *stiff* matrix (see (3.4.13) and (3.4.17) below) for each element and then form the whole matrix of the equation by summation of all the stiff matrices. The latter approach is more convenient and suitable by computer, especially for two-dimensional high order difference schemes and irregular meshes.

Take any a triangular element K_Q . Let $P_l(x_l, y_l)$ ($l = i, j, k$, counterclockwise) be the vertexes, M_i the midpoint of $\overline{P_jP_k}$, P_{ij} the point on $\overline{P_iP_j}$ such that $|\overline{P_iP_{ij}}| = \frac{1}{3}|\overline{P_iP_j}|$, Q_i the midpoint of $\overline{P_{ij}P_{ik}}$ etc. (cf. Fig. 3.4.3).

For $u_h \in U_h$, write $u_P = u_h(P)$. Then on K_Q

$$u_h = \sum_{l=i,j,k} (u_{P_l} \phi_{P_l} + u_{M_l} \phi_{M_l}).$$

Perform a linear transformation

$$\lambda_j = \frac{1}{2S_Q} \begin{vmatrix} 1 & x & y \\ 1 & x_k & y_k \\ 1 & x_i & y_i \end{vmatrix}, \quad \lambda_k = \frac{1}{2S_Q} \begin{vmatrix} 1 & x & y \\ 1 & x_i & y_i \\ 1 & x_j & y_j \end{vmatrix}. \quad (3.4.8)$$

Then K_Q is transformed into a reference element $\hat{K}_Q$ with vertexes $\hat{P}_i(0,0)$, $\hat{P}_j(1,0)$, $\hat{P}_k(0,1)$ on $\lambda_j\lambda_k$ plane; M_i, P_{ij}, Q_i become $\hat{M}_i, \hat{P}_{ij}$,

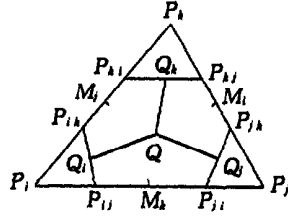


Fig. 3.4.3

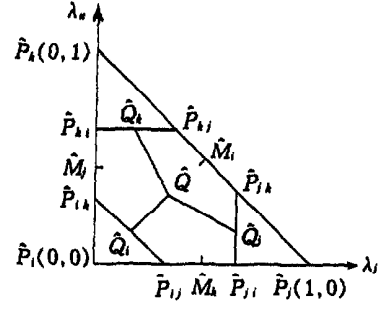


Fig. 3.4.4

$\hat{Q}_i$ etc. (cf. Fig. 3.4.4); and

$$\begin{aligned} u_h = & u_{P_i}(1 - \lambda_j - \lambda_k)(1 - 2\lambda_j - 2\lambda_k) + u_{P_j}(2\lambda_j - 1)\lambda_j \\ & + u_{P_k}(2\lambda_k - 1)\lambda_k + 4u_{M_i}\lambda_j\lambda_k \\ & + 4u_{M_j}(1 - \lambda_j - \lambda_k)\lambda_k + 4u_{M_k}(1 - \lambda_j - \lambda_k)\lambda_j. \end{aligned} \quad (3.4.9)$$

By (3.4.8) and (3.4.9) we have

$$\frac{\partial u_h}{\partial x} = \frac{1}{2S_Q} \left[\frac{\partial u_h}{\partial \lambda_j}(y_k - y_i) - \frac{\partial u_h}{\partial \lambda_k}(y_j - y_i) \right], \quad (3.4.10a)$$

$$\frac{\partial u_h}{\partial y} = \frac{1}{2S_Q} \left[-\frac{\partial u_h}{\partial \lambda_j}(x_k - x_i) + \frac{\partial u_h}{\partial \lambda_k}(x_j - x_i) \right], \quad (3.4.10b)$$

$$\begin{aligned} \frac{\partial u_h}{\partial \lambda_j} = & u_{P_i}(4\lambda_j + 4\lambda_k - 3) + u_{P_j}(4\lambda_j - 1) \\ & + 4u_{M_i}\lambda_k - 4u_{M_j}\lambda_k + 4u_{M_k}(1 - 2\lambda_j - \lambda_k), \end{aligned} \quad (3.4.11a)$$

$$\begin{aligned} \frac{\partial u_h}{\partial \lambda_k} = & u_{P_i}(4\lambda_j + 4\lambda_k - 3) + u_{P_k}(4\lambda_k - 1) \\ & + 4u_{M_i}\lambda_j + 4u_{M_j}(1 - \lambda_j - 2\lambda_k) - 4u_{M_k}\lambda_j. \end{aligned} \quad (3.4.11b)$$

The bilinear form $a(u_h, v_h)$ on $U_h \times V_h$ reads

$$a(u_h, v_h) = \sum_{K \in T_h} I_K(u_h, v_h), \quad (3.4.12)$$

where

$$\begin{aligned} I_K(u_h, v_h) = & \sum_{l=i,j,k} \left[v_{P_l} \int_{L_1} \left(-\frac{\partial u_h}{\partial x} dy + \frac{\partial u_h}{\partial y} dx \right) \right. \\ & \left. + v_{M_l} \int_{L_2} \left(-\frac{\partial u_h}{\partial x} dy + \frac{\partial u_h}{\partial y} dx \right) \right], \end{aligned} \quad (3.4.13)$$

where $L_1 = \overline{P_{l,l+1}P_{l,l+2}}$, $L_2 = \overline{P_{l+2,l+1}Q_{l+2}QQ_{l+1}P_{l+1,l+2}}$, $i+1=j$, $j+1=k$, $k+1=i$. It follows from (3.4.8) that

$$\begin{cases} d\lambda_j = \frac{1}{2S_Q} [(y_k - y_i)dx - (x_k - x_i)dy], \\ d\lambda_k = \frac{1}{2S_Q} [(y_j - y_i)dx - (x_j - x_i)dy], \end{cases} \quad (3.4.14)$$

$$\begin{cases} dx = (x_j - x_i)d\lambda_j + (x_k - x_i)d\lambda_k, \\ dy = (y_j - y_i)d\lambda_j + (y_k - y_i)d\lambda_k. \end{cases} \quad (3.4.15)$$

So

$$\begin{aligned} & \int_L \left(-\frac{\partial u_h}{\partial x} dy + \frac{\partial u_h}{\partial y} dx \right) \\ &= \frac{1}{2S_Q} \int_L \left\{ \left[-\frac{\partial u_h}{\partial \lambda_j} (y_k - y_i) + \frac{\partial u_h}{\partial \lambda_k} (y_j - y_i) \right] [(y_j - y_i)d\lambda_j \right. \\ & \quad \left. + (y_k - y_i)d\lambda_k] + \left[-\frac{\partial u_h}{\partial \lambda_j} (x_k - x_i) \right. \right. \\ & \quad \left. \left. + \frac{\partial u_h}{\partial \lambda_k} (x_j - x_i) \right] [(x_j - x_i)d\lambda_j + (x_k - x_i)d\lambda_k] \right\} \\ &= \frac{1}{2S_Q} \int_L \left[-a^2 \frac{\partial u_h}{\partial \lambda_j} d\lambda_k + b^2 \frac{\partial u_h}{\partial \lambda_k} d\lambda_j \right. \\ & \quad \left. + \frac{a^2 + b^2 - c^2}{2} \left(-\frac{\partial u_h}{\partial \lambda_j} d\lambda_j + \frac{\partial u_h}{\partial \lambda_k} d\lambda_k \right) \right], \end{aligned} \quad (3.4.16)$$

where $a = |P_i P_k|$, $b = |P_i P_j|$, $c = |P_j P_k|$ and $\hat{L}$ is the image of L by transformation (3.4.8). Using (3.4.16) to compute (3.4.13) results in

$$I_K(u_h, v_h) = \frac{1}{36S_Q} [v_{P_i}, v_{P_j}, v_{P_k}, v_{M_i}, v_{M_j}, v_{M_k}] A \cdot [v_{P_i}, v_{P_j}, v_{P_k}, v_{M_i}, v_{M_j}, v_{M_k}]^T, \quad (3.4.17)$$

where $A = [a_{ij}]$ is a 6×6 matrix with

$$\begin{aligned} a_{11} &= 10c^2, \quad a_{12} = a_{21} = a^2 - b^2 + c^2, \\ a_{13} &= a_{31} = -a^2 + b^2 + c^2, \quad a_{14} = -4c^2, \\ a_{15} &= a_{46} = 8a^2 - 8b^2 - 4c^2, \quad a_{16} = a_{45} = -8a^2 + 8b^2 - 4c^2, \\ a_{22} &= 10a^2, \quad a_{23} = a_{32} = a^2 + b^2 - c^2, \\ a_{24} &= a_{56} = -4a^2 - 8b^2 + 8c^2, \quad a_{25} = -4a^2, \\ a_{26} &= a_{54} = -4a^2 + 8b^2 - 8c^2, \quad a_{33} = 10b^2, \\ a_{34} &= a_{65} = -8a^2 - 4b^2 + 8c^2, \quad a_{35} = a_{64} = 8a^2 - 4b^2 - 8c^2, \\ a_{36} &= -4b^2, \quad a_{41} = -2c^2, \quad a_{42} = -5a^2 - 3b^2 + 3c^2, \\ a_{43} &= -3a^2 - 5b^2 + 3c^2, \quad a_{44} = 8a^2 + 8b^2 + 4c^2, \\ a_{51} &= 3a^2 - 3b^2 - 5c^2, \quad a_{52} = -2a^2, \quad a_{53} = 3a^2 - 5b^2 - 3c^2, \\ a_{55} &= 4a^2 + 8b^2 + 8c^2, \quad a_{61} = -3a^2 + 3b^2 - 5c^2, \\ a_{62} &= -5a^2 + 3b^2 - 3c^2, \quad a_{63} = -2b^2, \quad a_{66} = 8a^2 + 4b^2 + 8c^2. \end{aligned}$$

Here $\frac{1}{36S_Q} A$ is the stiff matrix of the element.

3.4.3 a priori estimates

Let us introduce the discrete semi- and full-norms:

$$||u_h||_{0,h} = \left(\sum_{K \in T_h} |u_h|_{0,h,K}^2 \right)^{1/2}, \quad (3.4.18)$$

$$|u_h|_{1,h} = \left(\sum_{K \in T_h} |u_h|_{1,h,K}^2 \right)^{1/2}, \quad (3.4.19)$$

$$||u_h||_{1,h} = (||u_h||_{0,h}^2 + |u_h|_{1,h}^2)^{1/2}, \quad (3.4.20)$$

where

$$\begin{aligned} |u_h|_{0,h,K} &= [(u_{P_i}^2 + u_{P_j}^2 + u_{P_k}^2 + u_{M_i}^2 + u_{M_j}^2 + u_{M_k}^2) S_Q / 6]^{1/2}, \\ |u_h|_{1,h,K} &= [(u_{P_i} - u_{M_i})^2 + (u_{P_j} - u_{M_j})^2 + (u_{P_k} - u_{M_k})^2 \\ &\quad + (u_{M_i} - u_{M_j})^2 + (u_{M_i} - u_{M_k})^2]^{1/2}. \end{aligned}$$

Lemma 3.4.1 *On the space U_h , $||\cdot||_{0,h}$ is equivalent with the L^2 -norm $||\cdot||_0$, and $|\cdot|_{1,h}$ is equivalent with the H^1 -semi-norm (and hence with the H_1 -norm $||\cdot||_1$), namely there exist constants c_i ($i = 1, 2, 3, 4$) independent of U_h such that*

$$c_1 ||u_h||_{0,h} \leq ||u_h||_0 \leq c_2 ||u_h||_{0,h}, \quad \forall u_h \in U_h, \quad (3.4.21)$$

$$c_3 |u_h|_{1,h} \leq |u_h|_1 \leq c_4 |u_h|_{1,h}, \quad \forall u_h \in U_h. \quad (3.4.22)$$

Proof For $u_h \in U_h$

$$||u_h||_0^2 = \sum_{K \in T_h} \int_K u_h^2 dx dy = \sum_{K \in T_h} 2S_Q \int_{\hat{K}} u_h^2 d\lambda_j d\lambda_k.$$

It is easy to show that $\int_{\hat{K}} u_h^2 d\lambda_j d\lambda_k$ is a positive definite bilinear form of $u_{P_i}, u_{P_j}, u_{P_k}, u_{M_i}, u_{M_j}, u_{M_k}$. Thus (3.4.21) holds.

Next let us turn to (3.4.22). Obviously we only have to prove the equivalence of $|\cdot|_{1,K}$ and $|\cdot|_{1,h,K}$. Since u_h is a quadratic polynomial on K , $(\frac{\partial u_h}{\partial x})^2 + (\frac{\partial u_h}{\partial y})^2$ is also a quadratic polynomial. Hence

$$\begin{aligned} |u_h|_{1,K}^2 &= \int_K \left[\left(\frac{\partial u_h}{\partial x} \right)^2 + \left(\frac{\partial u_h}{\partial y} \right)^2 \right] dx dy \\ &= \frac{1}{3} \sum_{l=i,j,k} \left[\left(\frac{\partial u_h(M_l)}{\partial x} \right)^2 + \left(\frac{\partial u_h(M_l)}{\partial y} \right)^2 \right] S_Q. \end{aligned} \quad (3.4.23)$$

It follows from (3.4.10) that

$$\begin{aligned} &\left(\frac{\partial u_h}{\partial x} \right)^2 + \left(\frac{\partial u_h}{\partial y} \right)^2 \\ &= \frac{1}{4S_Q} \left[a^2 \left(\frac{\partial u_h}{\partial \lambda_j} \right)^2 + b^2 \left(\frac{\partial u_h}{\partial \lambda_k} \right)^2 - 2ab \cos \angle P_j P_i P_k \frac{\partial u_h}{\partial \lambda_j} \frac{\partial u_h}{\partial \lambda_k} \right]. \end{aligned} \quad (3.4.24)$$

The quasi-uniformness of the decomposition leads to the existence of $\sigma > 0$ and $\theta_0 > 0$ satisfying

$$\frac{h_K}{\rho_K} \leq \sigma, \quad \theta_K \geq \theta_0, \quad \forall K \in T_h, \quad (3.4.25)$$

where ρ_K is the diameter of the inscribed circle of K , h_K the maximum side-length of K , and θ_K the minimum interior angle of K . By (3.4.24) and (3.4.25) we have

$$\begin{aligned} & \frac{1 - \cos \theta_0}{\sigma^2} \left[\left(\frac{\partial u_h}{\partial \lambda_j} \right)^2 + \left(\frac{\partial u_h}{\partial \lambda_k} \right)^2 \right] \\ & \leq \left[\left(\frac{\partial u_h}{\partial x} \right)^2 + \left(\frac{\partial u_h}{\partial y} \right)^2 \right] S_Q \leq 2\sigma^2 \left[\left(\frac{\partial u_h}{\partial \lambda_j} \right)^2 + \left(\frac{\partial u_h}{\partial \lambda_k} \right)^2 \right]. \end{aligned} \quad (3.4.26)$$

By (3.4.23) and (3.4.26) there exist constants $c'_3, c'_4 > 0$ such that

$$\begin{aligned} & c'_3 \sum_{l=i,j,k} \left[\left(\frac{\partial u_h(\hat{M}_l)}{\partial \lambda_j} \right)^2 + \left(\frac{\partial u_h(\hat{M}_l)}{\partial \lambda_k} \right)^2 \right] \\ & \leq |u_h|_{1,K}^2 \leq c'_4 \sum_{l=i,j,k} \left[\left(\frac{\partial u_h(\hat{M}_l)}{\partial \lambda_j} \right)^2 + \left(\frac{\partial u_h(\hat{M}_l)}{\partial \lambda_k} \right)^2 \right], \end{aligned} \quad (3.4.27)$$

where $\hat{M}_l$ is the image of M_l by the transformation (3.4.8). Write

$$\begin{cases} z_1 = u_{P_i} - u_{M_i}, & z_2 = u_{P_j} - u_{M_j}, \\ z_3 = u_{P_k} - u_{M_k}, & z_4 = u_{M_i} - u_{M_j}, \\ z_5 = u_{M_i} - u_{M_k}. \end{cases} \quad (3.4.28)$$

By (3.4.11) we have

$$\begin{aligned} & \sum_{l=i,j,k} \left[\left(\frac{\partial u_h(\hat{M}_l)}{\partial \lambda_j} \right)^2 + \left(\frac{\partial u_h(\hat{M}_l)}{\partial \lambda_k} \right)^2 \right] \\ & = (z_1 + z_2 + z_3 + 2z_5)^2 + (z_1 + z_3 + 2z_4 + z_5)^2 \\ & \quad + (-z_1 - z_2 + 3z_4 - 2z_5)^2 + (-z_1 + z_3 - z_5)^2 \\ & \quad + (-z_1 + z_2 - z_4)^2 + (-z_1 - z_3 + 3z_5 - 2z_4)^2. \end{aligned} \quad (3.4.29)$$

It is easy to check that the right-hand side of the above equation is a positive definite bilinear form of $z_1, z_2, \dots, z_5$, and hence it is equivalent to $\sum_{i=1}^5 z_i^2$. Now by (3.4.27) $|u_h|_{1,K}$ is equivalent to $|u_h|_{1,h,K}$. $\square$

Denote by $\Pi_h w$ and $\Pi_h^* w$ the interpolations of w in U_h and V_h respectively:

$$\Pi_h w = \sum_{P_0 \in \Omega_h} w(P_0) \phi_{P_0} + \sum_{M_0 \in \dot{M}_h} w(M_0) \phi_{M_0}, \quad (3.4.30a)$$

$$\Pi_h^* w = \sum_{P_0 \in \Omega_h} w(P_0) \psi_{P_0} + \sum_{M_0 \in \dot{M}_h} w(M_0) \psi_{M_0}, \quad (3.4.30b)$$

Theorem 3.4.1 *Suppose that the maximum angle of each element of the triangulation T_h is not greater than $\frac{\pi}{2}$, and that the ratio τ of the lengths of the two sides of the maximum angle satisfies $\tau \in [\sqrt{\frac{2}{3}}, \sqrt{\frac{3}{2}}]$. Then there exists a constant $\alpha > 0$ independent of U_h such that*

$$a(u_h, \Pi_h^* u_h) \geq \alpha \|u_h\|_1^2, \quad \forall u_h \in U_h. \quad (3.4.31)$$

Proof By (3.4.17)

$$I_K(u_h, \Pi_h^* u_h) = \frac{1}{36S_Q} Z^T \tilde{A} Z, \quad (3.4.32)$$

where $Z = [z_1, z_2, z_3, z_4, z_5]^T$, and $\tilde{A} = [\tilde{a}_{ij}]$ is a symmetric 5×5 matrix with

$$\begin{aligned} \tilde{a}_{11} &= 10c^2, & \tilde{a}_{12} &= a^2 - b^2 + c^2, \\ \tilde{a}_{13} &= -a^2 + b^2 + c^2, & \tilde{a}_{14} &= \frac{-13a^2 + 13b^2 + 7c^2}{2}, \\ \tilde{a}_{15} &= \frac{13a^2 - 13b^2 + 7c^2}{2}, & \tilde{a}_{22} &= 10a^2, \\ \tilde{a}_{23} &= a^2 + b^2 - c^2, & \tilde{a}_{24} &= -7a^2, \end{aligned}$$

$$\begin{aligned}
\tilde{a}_{25} &= \frac{7a^2 - 13b^2 + 13c^2}{2}, & \tilde{a}_{33} &= 10b^2, \\
\tilde{a}_{34} &= \frac{-13a^2 + 7b^2 + 13c^2}{2}, & \tilde{a}_{35} &= -7b^2, \\
\tilde{a}_{44} &= 8(a^2 + b^2 + c^2), & \tilde{a}_{45} &= -4(a^2 + b^2 + c^2), \\
\tilde{a}_{55} &= 8(a^2 + b^2 + c^2).
\end{aligned}$$

Next we prove the existence of a constant $\tilde{\alpha} > 0$ independent of K such that

$$I_K(u_h, \Pi_h^* u_h) \geq \tilde{\alpha} Z^T Z = \tilde{\alpha} |u|_{1,h,K}^2, \quad \forall u_h \in U_h. \quad (3.4.33)$$

Define

$$B = [b_{ij}] = G^T \tilde{A} G,$$

where

$$G = \begin{bmatrix} 1 & 0 & 0 & -5/8 & 3/8 \\ 0 & 1 & 0 & 3/8 & -5/8 \\ 0 & 0 & 1 & 3/8 & 3/8 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then B is a symmetric 5×5 matrix with

$$\begin{aligned}
b_{11} &= 10c^2, \quad b_{12} = -b_{13} = a^2 - b^2 + c^2, \quad b_{14} = \frac{3}{2}c^2, \\
b_{15} &= \frac{11}{2}a^2 - \frac{11}{2}b^2, \quad b_{22} = 10a^2, \quad b_{23} = a^2 + b^2 - c^2, \\
b_{24} &= -\frac{11}{2}b^2 + \frac{11}{2}c^2, \quad b_{25} = \frac{3}{2}a^2, \quad b_{33} = 10b^2, \\
b_{34} &= -b_{35} = -\frac{11}{2}a^2 + \frac{11}{2}c^2, \\
b_{44} &= \frac{35}{16}a^2 + \frac{35}{16}b^2 + \frac{187}{16}c^2, \\
b_{45} &= -\frac{81}{16}a^2 + \frac{35}{16}b^2 - \frac{81}{16}c^2, \\
b_{55} &= \frac{187}{16}a^2 + \frac{35}{16}b^2 + \frac{35}{16}c^2.
\end{aligned}$$

Without loss of generality, we assume b is the largest side of K . Then

$$\tau_0 \leq \frac{a}{c} \leq \tau_0^{-1}, \quad \tau_0 = \sqrt{\frac{2}{3}}.$$

The matrix B is positive definite because

$$\begin{aligned} b_{11} - \sum_{i \neq 1} |b_{1i}| &= \frac{13}{2}c^2 - \left| \frac{11}{2}a^2 - \frac{11}{2}b^2 \right| \geq c^2 \geq \tau_0 ac \sin \theta = 2\tau_0 S_Q, \\ b_{22} - \sum_{i \neq 2} |b_{2i}| &= \frac{13}{2}a^2 - \left| -\frac{11}{2}b^2 + \frac{11}{2}c^2 \right| \geq 2\tau_0 S_Q, \\ b_{33} - \sum_{i \neq 3} |b_{3i}| &= 8b^2 - |11a^2 - 11c^2| \geq 2\tau_0(11\tau_0 - 3)S_Q, \\ b_{44} - \sum_{i \neq 4} |b_{4i}| &= -\frac{46}{16}a^2 - \frac{18}{16}b^2 + \frac{170}{16}c^2 - \left| \frac{11}{2}a^2 + \frac{11}{2}c^2 \right| \geq \tau_0 S_Q, \\ b_{55} - \sum_{i \neq 5} |b_{5i}| &= \frac{170}{16}a^2 - \frac{18}{16}b^2 - \frac{46}{16}c^2 - \left| \frac{11}{2}a^2 - \frac{11}{2}b^2 \right| \geq \tau_0 S_Q. \end{aligned}$$

By the Gerschgorin theorem the smallest eigenvalue

$$\lambda_{\min} \geq \tau_0 S_Q. \quad (3.4.34)$$

Therefore, by (3.4.32) there exists a constant $\tilde{\alpha} > 0$ such that

$$\begin{aligned} I_K(u_h, \Pi_h^* u_h) &\geq \frac{1}{36S_Q} (G^{-1}Z)^T B (G^{-1}Z) \\ &\geq \frac{\tau_0}{36} (G^{-1}Z)^T (G^{-1}Z) \geq \tilde{\alpha} Z^T Z = \tilde{\alpha} |u|_{1,h,K}^2, \end{aligned} \quad (3.4.35)$$

which gives (3.4.33). Combining (3.4.12), (3.4.19) and Lemma 3.4.1 leads to (3.4.31). $\square$

3.4.4 Error estimates

Theorem 3.4.2 *Suppose that the triangulation T_h satisfies the conditions in Theorem 3.4.1. Let u be the solution to (3.4.2) and u_h to the quadratic element difference scheme (3.4.6). If $u \in H^3(\Omega)$, then the following error estimate holds:*

$$\|u - u_h\|_1 \leq Ch^2 |u|_3. \quad (3.4.36)$$

Proof The proof is similar to that of Theorem 3.2.2. (3.4.2), (3.4.6) and the *a priori* estimate (3.4.31) imply

$$\|u - u_h\|_1 \leq \|u - \Pi_h u\|_1 + \frac{1}{\alpha} \sup_{\bar{u}_h \in U_h} \frac{|a(u - \Pi_h u, \Pi_h^* \bar{u}_h)|}{\|\bar{u}_h\|_1}. \quad (3.4.37)$$

By (3.4.12) and (3.4.13) we have

$$a(u - \Pi_h u, \Pi_h^* \bar{u}_h) = \sum_{K \in T_h} I_K(u - \Pi_h u, \Pi_h^* \bar{u}_h), \quad (3.4.38)$$

$$\begin{aligned} & I_K(u - \Pi_h u, \Pi_h^* \bar{u}_h) \\ &= \sum_{l=i,j,k} \left[\int_{Q_l P_{l,i+1}} \left(-\frac{\partial(u - \Pi_h u)}{\partial x} dy + \frac{\partial(u - \Pi_h u)}{\partial y} dx \right) (\bar{u}_{M_{l+2}} - \bar{u}_{P_l}) \right. \\ & \quad + \int_{Q_l P_{l,i+2}} \left(-\frac{\partial(u - \Pi_h u)}{\partial x} dy + \frac{\partial(u - \Pi_h u)}{\partial y} dx \right) (\bar{u}_{P_l} - \bar{u}_{M_{l+1}}) \\ & \quad \left. + \int_{QQ_l} \left(-\frac{\partial(u - \Pi_h u)}{\partial x} dy + \frac{\partial(u - \Pi_h u)}{\partial y} dx \right) (\bar{u}_{M_{l+2}} - \bar{u}_{M_{l+1}}) \right], \end{aligned} \quad (3.4.39)$$

where $i+1 = j, j+1 = k, k+1 = i, \bar{u}_P = \bar{u}_h(P)$ etc. By the definition of $|\cdot|_{1,h,K}$ we have

$$|\bar{u}_{M_{l+2}} - \bar{u}_{P_l}|, |\bar{u}_{P_l} - \bar{u}_{M_{l+1}}|, |\bar{u}_{M_{l+2}} - \bar{u}_{M_{l+1}}| \leq C |\bar{u}_h|_{1,h,K}. \quad (3.4.40)$$

Let $L = \overline{Q_l P_{l,i+1}}$ (or $\overline{Q_l P_{l,i+2}}, \overline{QQ_l}$) and $\phi_1 = \frac{\partial(u - \Pi_h u)}{\partial x}$, $\phi_2 = \frac{\partial(u - \Pi_h u)}{\partial y}$. Then

$$\begin{aligned} & \left| \int_L \left(-\frac{\partial(u - \Pi_h u)}{\partial x} dy + \frac{\partial(u - \Pi_h u)}{\partial y} dx \right) \right| \\ & \leq \int_L (|\phi_1| + |\phi_2|) ds \leq h^{1/2} \left[\int_L (\phi_1^2 + \phi_2^2) ds \right]^{1/2}. \end{aligned} \quad (3.4.41)$$

Assume that the linear mapping (3.4.8) maps the element K onto the reference element $\hat{K}$, the segment L into $\hat{L}$, and the function ϕ_i on K into the function $\hat{\phi}_i(\lambda_j, \lambda_k) = \phi_i(x, y)$, ($i = 1, 2$) on $\hat{K}$. Then we have

$$\int_L \phi_i^2 ds \leq h \int_{\hat{L}} \hat{\phi}_i^2 d\hat{s}, \quad i = 1, 2. \quad (3.4.42)$$

Let L be a part of the boundary of $K_{M_i}^*$. Employing the trace theorem on $\hat{K}_{M_i}^* \cap \hat{K}$ we have a constant $C > 0$ independent of K such that

$$\int_L \hat{\phi}_i^2 d\hat{s} \leq C \|\hat{\phi}_i\|_{1,\hat{K}}^2, \quad i = 1, 2. \quad (3.4.43)$$

After the affine transformation, the Sobolev semi-norms have the following relationships:

$$\begin{cases} |\hat{\phi}_i|_{0,\hat{K}} \leq Ch^{-1} |\phi_i|_{0,K}, \\ |\hat{\phi}_i|_{1,\hat{K}} \leq C |\phi_i|_{1,K}, \end{cases} \quad i = 1, 2. \quad (3.4.44)$$

By (3.4.42)-(3.4.44) and the interpolation approximation theorem we have

$$\begin{aligned} \int_L \phi_i^2 ds &\leq Ch(h^{-1} |\phi_i|_{0,K} + |\phi_i|_{1,K})^2 \\ &\leq Ch(h^{-1} |u - \Pi_h u|_{1,K} + |u - \Pi_h u|_{2,K})^2 \\ &\leq Ch^3 |u|_{3,K}^2. \end{aligned} \quad (3.4.45)$$

A combination of (3.4.39)-(3.4.41) and (3.4.45) yields

$$I_K(u - \Pi_h u, \Pi_h^* \bar{u}_h) \leq Ch^2 |u|_{3,K} |\bar{u}_h|_{1,h,K}. \quad (3.4.46)$$

This together with (3.4.38) and Lemma 3.4.1 leads to

$$a(u - \Pi_h u, \Pi_h^* \bar{u}_h) \leq Ch^2 |u|_3 |\bar{u}_h|_1. \quad (3.4.47)$$

Finally, (3.4.36) results from (3.4.37), (3.4.47) and the interpolation approximation property. $\square$

3.4.5 Numerical example

The following problem is approximated by the five point finite difference method (FDM), the quadratic finite element method (FEM) and the quadratic element generalized difference method (GDM), respectively:

$$\begin{cases} -\Delta u = 2 \sin x \sin y, & \text{on } \Omega = (0, \pi) \times (0, \pi), \\ u|_{\partial\Omega} = 0. \end{cases} \quad (3.4.48)$$

Place a right triangular decomposition on Ω (see Fig. 3.4.5) with the right-angle-side length $h = \frac{\pi}{N}$, $x_i = ih, y_j = jh$, $i, j = 1, 2, \dots, N$. The results of the three methods as well as the true solution (TS) $u(x, y) = \sin x \sin y$ are given in Table 3.4.1.

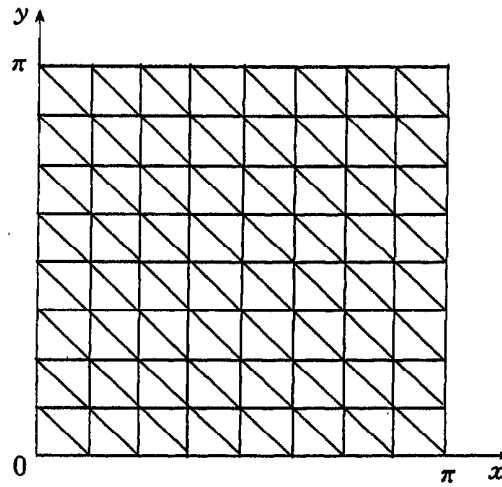


Fig. 3.4.5

Table 3.4.1. Numerical results ([A-41])

	FDM(N=8)	FEM(N=8)	GDM(N=8)	TS
(x_1, y_1)	0.148343	0.146418	0.146178	0.146447
(x_2, y_1)	0.274102	0.270546	0.271157	0.270598
(x_2, y_2)	0.506475	0.499904	0.502044	0.500000
(x_3, y_1)	0.358132	0.353485	0.355223	0.353553
(x_3, y_2)	0.661742	0.653156	0.656951	0.653281
(x_3, y_3)	0.864607	0.853389	0.858948	0.853553
(x_4, y_1)	0.387639	0.382610	0.385339	0.382683
(x_4, y_2)	0.716264	0.706971	0.712014	0.707107
(x_4, y_3)	0.935844	0.923702	0.930291	0.923879
(x_4, y_4)	1.012950	0.999808	1.006940	1.000000

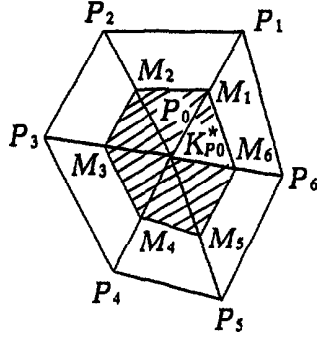


Fig. 3.5.1

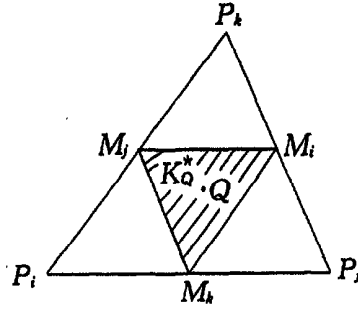


Fig. 3.5.2

3.5 Cubic Element Difference Schemes

In this section we discuss a generalized difference scheme based on a cubic element of Hermite type for problem (3.4.1).

3.5.1 Trial and test function spaces

Let T_h be a quasi-uniform triangulation of $\bar{\Omega}$ as in §3.2 and h the largest side length of all the triangles. T_h consists of a finite number of triangular elements K_Q 's, Q being the barycenter of the triangle. Denote by $\bar{\Omega}_h$ and Ω_h^* the sets of the vertexes and the barycenters of all the triangular elements, respectively, and $\dot{\Omega}_h = \bar{\Omega}_h \setminus \partial\Omega$. Let S_Q be the area of K_Q .

For the dual decomposition, we consider a vertex P_0 of a triangular element. Suppose P_i ($i = 1, 2, \dots, 6$) are the adjacent vertexes of P_0 and M_i is the midpoint of $\overline{P_0P_i}$ (cf. Fig. 3.5.1). Connect M_i successively to obtain a polygonal region $K_{P_0}^*$ surrounding P_0 , as an element of the dual decomposition. Suppose that Q is the barycenter of a triangular element $K = \triangle P_iP_jP_k$, and that M_i, M_j, M_k are the midpoints of $\overline{P_jP_k}, \overline{P_kP_i}, \overline{P_iP_j}$ respectively (cf. Fig. 3.5.2). Connecting M_i, M_j and M_k results in a triangular region K_Q^* surrounding Q , which is also taken as an element of the dual decomposition. These

two kinds of dual elements form a dual decomposition, denoted by T_h^* .

The trial function space is chosen as the Hermitian cubic element space related to T_h . There are three basis functions corresponding to a node $P_0 \in \bar{\Omega}_h$, denoted by $\phi_{P_0}^{(k)}$ ($k = 0, 1, 2$) and satisfying the following interpolation conditions:

$$\begin{cases} \phi_{P_0}^{(0)}(P_0) = 1, \\ \phi_{P_0}^{(0)}(P) = 0, \text{ if } P \in \bar{\Omega}_h \cup \Omega_h^* \setminus \{P_0\}, \\ \frac{\partial}{\partial x} \phi_{P_0}^{(0)}(P) = \frac{\partial}{\partial y} \phi_{P_0}^{(0)}(P) = 0, \text{ if } P \in \bar{\Omega}_h; \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial x} \phi_{P_0}^{(1)}(P_0) = 1, \\ \frac{\partial}{\partial x} \phi_{P_0}^{(1)}(P) = 0, \text{ if } P \in \bar{\Omega}_h \setminus \{P_0\}, \\ \phi_{P_0}^{(1)}(P) = 0, \text{ if } P \in \bar{\Omega}_h \cup \Omega_h^*, \\ \frac{\partial}{\partial y} \phi_{P_0}^{(1)}(P) = 0, \text{ if } P \in \bar{\Omega}_h; \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial y} \phi_{P_0}^{(2)}(P_0) = 1, \\ \frac{\partial}{\partial y} \phi_{P_0}^{(2)}(P) = 0, \text{ if } P \in \bar{\Omega}_h \setminus \{P_0\}, \\ \phi_{P_0}^{(2)}(P) = 0, \text{ if } P \in \bar{\Omega}_h \cup \Omega_h^*, \\ \frac{\partial}{\partial x} \phi_{P_0}^{(2)}(P) = 0, \text{ if } P \in \bar{\Omega}_h. \end{cases}$$

There is also a basis function $\phi_{Q_0}(P)$ related to the barycenter Q of the triangular element, satisfying the interpolation condition

$$\begin{cases} \phi_{Q_0}(Q_0) = 1, \\ \phi_{Q_0}(P) = 0, \text{ if } P \in \bar{\Omega}_h \cup \Omega_h^* \setminus \{Q_0\}, \\ \frac{\partial}{\partial x} \phi_{Q_0}(P) = \frac{\partial}{\partial y} \phi_{Q_0}(P) = 0, \text{ if } P \in \bar{\Omega}_h. \end{cases}$$

Taking into account of the boundary condition $u|_{\partial\Omega} = 0$, we choose

$$U_h = \text{span}\{\phi_{P_0}^{(k)}, \phi_{Q_0} : P_0 \in \dot{\Omega}_h, k = 0; P_0 \in \bar{\Omega}_h, k = 1, 2; Q_0 \in \Omega_h^*\}.$$

The test function space is chosen as the piecewise constant and piecewise linear function space. The three basis functions related to $P_0 = (x_0, y_0) \in \bar{\Omega}_h$ are

$$\begin{aligned}\psi_{P_0}^{(0)}(P) &= \begin{cases} 1, & \text{if } P \in K_{P_0}^*, \\ 0, & \text{if } P \notin K_{P_0}^*; \end{cases} \\ \psi_{P_0}^{(1)}(P) &= \begin{cases} x - x_0, & \text{if } P \in K_{P_0}^*, \\ 0, & \text{if } P \notin K_{P_0}^*; \end{cases} \\ \psi_{P_0}^{(2)}(P) &= \begin{cases} y - y_0, & \text{if } P \in K_{P_0}^*, \\ 0, & \text{if } P \notin K_{P_0}^*. \end{cases}\end{aligned}$$

The basis function related to $Q_0 \in \Omega_h^*$ is

$$\psi_{Q_0}^{(0)}(P) = \begin{cases} 1, & \text{if } P \in K_{Q_0}^*, \\ 0, & \text{if } P \notin K_{Q_0}^*. \end{cases}$$

Similarly we require that the functions in V_h vanish on the boundary. So we have

$$V_h = \text{span}\{\psi_{P_0}^{(k)}, \psi_{Q_0} : P_0 \in \dot{\Omega}_h, k = 0; P_0 \in \bar{\Omega}_h, k = 1, 2; Q_0 \in \Omega_h^*\}.$$

3.5.2 Generalized difference equations

The cubic element difference scheme corresponding to U_h and V_h defined above is: Find $u_h \in U_h$ satisfying

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad (3.5.1)$$

or equivalently

$$\begin{cases} a(u_h, \psi_{P_0}^{(k)}) = (f, \psi_{P_0}^{(k)}), & (3.5.2a) \\ P_0 \in \dot{\Omega}_h, k = 0; P_0 \in \bar{\Omega}_h, k = 1, 2, \\ a(u_h, \psi_{Q_0}) = (f, \psi_{Q_0}), Q_0 \in \Omega_h^*, & (3.5.2b) \end{cases}$$

where

$$a(u_h, \psi_{P_0}^{(0)}) = \int_{\partial K_{P_0}^*} \frac{\partial u_h}{\partial y} dx - \frac{\partial u_h}{\partial x} dy, \quad (3.5.3a)$$

$$a(u_h, \psi_{P_0}^{(1)}) = \int_{K_{P_0}^*} \frac{\partial u_h}{\partial x} dx dy + \int_{\partial K_{P_0}^*} \frac{\partial u_h}{\partial y} \psi_{P_0}^{(1)} dx - \frac{\partial u_h}{\partial x} \psi_{P_0}^{(1)} dy, \quad (3.5.3b)$$

$$a(u_h, \psi_{P_0}^{(2)}) = \int_{K_{P_0}^*} \frac{\partial u_h}{\partial y} dx dy + \int_{\partial K_{P_0}^*} \frac{\partial u_h}{\partial y} \psi_{P_0}^{(2)} dx - \frac{\partial u_h}{\partial x} \psi_{P_0}^{(2)} dy, \quad (3.5.3c)$$

$$a(u_h, \psi_{Q_0}) = \int_{\partial K_{Q_0}^*} \frac{\partial u_h}{\partial y} dx - \frac{\partial u_h}{\partial x} dy. \quad (3.5.3d)$$

Define an interpolation operator $\Pi_h^* : U_h \rightarrow V_h$ by

$$\begin{aligned} \Pi_h^* \bar{u}_h = & \sum_{P_0 \in \bar{\Omega}_h} \left[\bar{u}_h(P_0) \psi_{P_0}^{(0)} + \frac{\partial \bar{u}_h(P_0)}{\partial x} \psi_{P_0}^{(1)} \right. \\ & \left. + \frac{\partial \bar{u}_h(P_0)}{\partial y} \psi_{P_0}^{(2)} \right] + \sum_{Q_0 \in \Omega_h^*} \bar{u}_h(Q_0) \psi_{Q_0}. \end{aligned} \quad (3.5.4)$$

Then, (3.5.1) is equivalent to

$$a(u_h, \Pi_h^* \bar{u}_h) = (f, \Pi_h^* \bar{u}_h), \quad \forall \bar{u}_h \in U_h, \quad (3.5.5)$$

To compute the element stiff matrix we write $a(u_h, \Pi_h^* \bar{u}_h)$ as

$$a(u_h, \Pi_h^* \bar{u}_h) = \sum_{K \in T_h} I_K(u_h, \Pi_h^* \bar{u}_h), \quad (3.5.6)$$

where $K = \triangle P_i P_j P_k$ (cf. Fig. 3.5.2),

$$\begin{aligned} & I_K(u_h, \Pi_h^* \bar{u}_h) \\ = & \sum_{l=i,j,k} \left[\bar{u}_h(P_l) I_K(u_h, \psi_{P_l}^{(0)}) + \frac{\partial \bar{u}_h(P_l)}{\partial x} I_K(u_h, \psi_{P_l}^{(1)}) \right. \\ & \left. + \frac{\partial \bar{u}_h(P_l)}{\partial y} I_K(u_h, \psi_{P_l}^{(2)}) \right] + \bar{u}_h(Q) I_K(u_h, \psi_Q), \end{aligned} \quad (3.5.7)$$

and each $I_K(u_h, \psi_P)$ is obtained by changing the integral regions K_P^* and ∂K_P^* in $a(u_h, \psi_P)$ (cf. (3.5.3)) into $K_P^* \cap K$ and $\partial K_P^* \cap K$ respectively.

On K , $u_h \in U_h$ can be expressed in terms of $(\lambda_i, \lambda_j, \lambda_k)$ as (cf. Example 3 in §1.1.4):

$$\begin{aligned} u_h = & 27\lambda_i\lambda_j\lambda_k u_h(Q) \\ & + \sum_{l=i,j,k} (-2\lambda_l^3 + 3\lambda_l^2 - 7\lambda_i\lambda_j\lambda_k) u_h(P_l) \\ & + \sum_{l=j,k} \left[(2\lambda_i\lambda_j\lambda_k - \lambda_l\lambda_j\lambda_k - \lambda_l^2\lambda_i) \frac{\partial u_h(P_l)}{\partial \lambda_l} \right. \\ & \left. + \sum_{\substack{m=i,j,k \\ m \neq l}} (\lambda_m^2\lambda_l - \lambda_i\lambda_j\lambda_k) \frac{\partial u_h(P_m)}{\partial \lambda_l} \right]. \end{aligned} \quad (3.5.8)$$

After the transformation $(x, y) \rightarrow (\lambda_j, \lambda_k)$, the triangular element K becomes the reference element $\hat{K}$ (cf. Fig. 3.5.3), and the points P_l, M_l and Q become $\hat{P}_l, \hat{M}_l$ and $\hat{Q}$ ($l = i, j, k$) respectively. Also note

$$\frac{\partial u_h}{\partial x} = \frac{1}{2S_Q} \left[(y_k - y_i) \frac{\partial u_h}{\partial \lambda_j} + (y_i - y_j) \frac{\partial u_h}{\partial \lambda_k} \right], \quad (3.5.9a)$$

$$\frac{\partial u_h}{\partial y} = \frac{1}{2S_Q} \left[(x_i - x_k) \frac{\partial u_h}{\partial \lambda_j} + (x_j - x_i) \frac{\partial u_h}{\partial \lambda_k} \right], \quad (3.5.9b)$$

$$dx = (x_j - x_i)d\lambda_j + (x_k - x_i)d\lambda_k, \quad (3.5.10a)$$

$$dy = (y_j - y_i)d\lambda_j + (y_k - y_i)d\lambda_k. \quad (3.5.10b)$$

The integrals in $I_K(u_h, \Pi_h^* \bar{u}_h)$ can be changed into the integrals on the reference element $\hat{K}$ on (λ_j, λ_k) plane by the transformation $(x, y) \rightarrow (\lambda_j, \lambda_k)$. Then the element stiff matrix can be obtained by computing $I_K(u_h, \Pi_h^* \bar{u}_h)$ as in §3.4.2.

3.5.3 *a priori* estimates

Let us introduce a discrete semi-norm

$$|u_h|_{1,h} = \left(\sum_{K \in T_h} |u_h|_{1,h,K}^2 \right)^{1/2}, \quad (3.5.11)$$

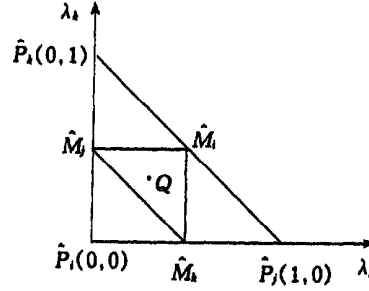


Fig. 3.5.3

where

$$|u_h|_{1,h,K}^2 = \sum_{l=i,j,k} \left[(u_h(P_l) - u_h(Q))^2 + \left(\frac{\partial u_h(P_l)}{\partial \lambda_j} \right)^2 + \left(\frac{\partial u_h(P_l)}{\partial \lambda_k} \right)^2 \right].$$

Lemma 3.5.1 *The discrete semi-norm $|\cdot|_{1,h}$ is equivalent to the H^1 -semi-norm $|\cdot|_1$ on U_h , namely there exist constants $c_1, c_2 > 0$ independent of U_h such that*

$$c_1 |u_h|_{1,h} \leq |u_h|_1 \leq c_2 |u_h|_{1,h}, \quad \forall u_h \in U_h. \quad (3.5.12)$$

Proof For $u_h \in U_h$,

$$\begin{aligned} |u_h|_1^2 &= \sum_{K \in T_h} \int_K \left[\left(\frac{\partial u_h}{\partial x} \right)^2 + \left(\frac{\partial u_h}{\partial y} \right)^2 \right] dx dy \\ &= \sum_{K \in T_h} \int_K \left[\left(\frac{\partial u_h}{\partial x} \right)^2 + \left(\frac{\partial u_h}{\partial y} \right)^2 \right] \cdot 2S_Q d\lambda_j d\lambda_k. \end{aligned} \quad (3.5.13)$$

It follows from (3.5.9) that

$$\begin{aligned} \left(\frac{\partial u_h}{\partial x} \right)^2 + \left(\frac{\partial u_h}{\partial y} \right)^2 &= \frac{1}{4S_Q^2} \left[a^2 \left(\frac{\partial u_h}{\partial \lambda_j} \right)^2 + b^2 \left(\frac{\partial u_h}{\partial \lambda_k} \right)^2 \right. \\ &\quad \left. - 2ab \cos \angle P_j P_i P_k \frac{\partial u_h}{\partial \lambda_j} \frac{\partial u_h}{\partial \lambda_k} \right], \end{aligned} \quad (3.5.14)$$

where $a = |\overline{P_k P_i}|$, $b = |\overline{P_i P_j}|$ and $c = |\overline{P_j P_k}|$. The regularity of the decomposition implies the existence of constants $\sigma > 0$ and $\theta_0 > 0$

such that

$$\frac{h_K}{\rho_K} \leq \sigma, \quad \theta_K \geq \theta_0, \quad \forall K \in T_h,$$

where ρ_K is the diameter of the inscribed circle of K , h_K the maximum side length of K and θ_K the minimum interior angle. By (3.5.14)

$$\begin{aligned} & \frac{1 - \cos \theta_0}{\sigma^2} \left[\left(\frac{\partial u_h}{\partial \lambda_j} \right)^2 + \left(\frac{\partial u_h}{\partial \lambda_k} \right)^2 \right] \\ & \leq \left[\left(\frac{\partial u_h}{\partial x} \right)^2 + \left(\frac{\partial u_h}{\partial y} \right)^2 \right] S_Q \leq 2\sigma^2 \left[\left(\frac{\partial u_h}{\partial \lambda_j} \right)^2 + \left(\frac{\partial u_h}{\partial \lambda_k} \right)^2 \right]. \end{aligned} \quad (3.5.15)$$

It easily follows from (3.5.8) that $\int_K \left[\left(\frac{\partial u_h}{\partial \lambda_j} \right)^2 + \left(\frac{\partial u_h}{\partial \lambda_k} \right)^2 \right] d\lambda_j d\lambda_k$ is a positive definite bilinear form of

$$\begin{aligned} \dot{Z} = & \left[u_h(P_j) - u_h(Q), u_h(P_k) - u_h(Q), u_h(P_i) - u_h(Q), \right. \\ & \left. \frac{\partial u_h(P_j)}{\partial \lambda_j}, \frac{\partial u_h(P_k)}{\partial \lambda_j}, \frac{\partial u_h(P_i)}{\partial \lambda_j}, \frac{\partial u_h(P_j)}{\partial \lambda_k}, \frac{\partial u_h(P_k)}{\partial \lambda_k}, \frac{\partial u_h(P_i)}{\partial \lambda_k} \right]^T. \end{aligned} \quad (3.5.16)$$

This together with (3.5.13) and (3.5.15) leads to the desired conclusion and completes the proof. $\square$

The norm $|\cdot|_{1,h}$ is also equivalent to the H^1 -norm $\|\cdot\|_1$ on U_h since $U_h \subset H_0^1(\Omega)$ and $|\cdot|_1$ is an equivalent norm on $H_0^1(\Omega)$.

Theorem 3.5.1 *Assume that the maximum angle of each element of the triangulation T_h is not greater than $\frac{\pi}{2}$ and that the ratio τ of the two side lengths of the maximum angle satisfies $\tau \in [\sqrt{\frac{2}{3}}, \sqrt{\frac{3}{2}}]$. Then there exists a constant $\alpha > 0$ independent of U_h such that*

$$a(u_h, \Pi_h^* u_h) \geq \alpha \|u_h\|_1^2, \quad \forall u_h \in U_h. \quad (3.5.17)$$

Proof Write $I_K(u_h, \Pi_h^* u_h)$ into a symmetric form

$$I_K(u_h, \Pi_h^* u_h) = \frac{1}{768 S_Q} Z^T A Z, \quad (3.5.18)$$

where A is a symmetric matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^T & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{bmatrix},$$

$$A_{11} =$$

$$\begin{bmatrix} 400a^2 & 96c^2 - 40a^2 - 40b^2 & 96b^2 - 40a^2 - 40c^2 \\ 96c^2 - 40a^2 - 40b^2 & 400b^2 & 96a^2 - 40b^2 - 40c^2 \\ 96b^2 - 40a^2 - 40c^2 & 96a^2 - 40b^2 - 40c^2 & 400c^2 \end{bmatrix},$$

$$A_{12} =$$

$$\begin{bmatrix} -96a^2 & 41c^2 - 7a^2 - 20b^2 & 41b^2 - 7a^2 - 20c^2 \\ 13a^2 + 11b^2 - 27c^2 & 41c^2 - 41a^2 + 48b^2 & 7c^2 - 14a^2 - 4b^2 \\ 13a^2 - 27b^2 + 11c^2 & 7b^2 - 14a^2 - 4c^2 & 41b^2 - 41a^2 + 48c^2 \end{bmatrix},$$

$$A_{13} =$$

$$\begin{bmatrix} 48a^2 - 41b^2 + 41c^2 & 11a^2 + 13b^2 - 27c^2 & 7c^2 - 4a^2 - 14b^2 \\ 41c^2 - 20a^2 - 7b^2 & -96b^2 & 41a^2 - 7b^2 - 20c^2 \\ 7a^2 - 14b^2 - 4c^2 & 13b^2 - 27a^2 + 11c^2 & 41a^2 - 41b^2 + 48c^2 \end{bmatrix},$$

$$A_{22} =$$

$$\begin{bmatrix} 34a^2 & a^2 + 5b^2 - 11c^2 & a^2 - 11b^2 + 5c^2 \\ a^2 + 5b^2 - 11c^2 & 7a^2 + 7b^2 + 15c^2 & -2a^2 + b^2 + c^2 \\ a^2 - 11b^2 + 5c^2 & -2a^2 + b^2 + c^2 & 7a^2 + 15b^2 + 7c^2 \end{bmatrix},$$

$$A_{23} =$$

$$\begin{bmatrix} -17a^2 + 4b^2 - 4c^2 & -3a^2 - 3b^2 + 7c^2 & 2a^2 + 4b^2 - 2c^2 \\ -2a^2 - 2b^2 + 13c^2 & 4a^2 - 17b^2 - 4c^2 & -2a^2 + b^2 - 3c^2 \\ a^2 - 2b^2 - 3c^2 & 4a^2 - 2b^2 - 2c^2 & -11a^2 - 11b^2 + 10c^2 \end{bmatrix},$$

$$A_{33} =$$

$$\begin{bmatrix} 7a^2 + 7b^2 + 15c^2 & 5a^2 + b^2 - 11c^2 & a^2 - 2b^2 + c^2 \\ 5a^2 + b^2 - 11c^2 & 34b^2 & -11a^2 + b^2 + 5c^2 \\ a^2 - 2b^2 + c^2 & -11a^2 + b^2 + 5c^2 & 15a^2 + 7b^2 + 7c^2 \end{bmatrix}.$$

Let the shortest and the longest sides of K are $\overline{P_i P_j}$ and $\overline{P_j P_k}$ respectively. Divide our discussion into the following two cases.

Case 1. Suppose $c^2 \leq \frac{3}{4}(a^2 + b^2)$. Take $a^2 = b^2 = c^2 = 1$ in the subblocks A_{ij} of A to obtain $\hat{A}_{ij}$. Set

$$G = \begin{bmatrix} \frac{1}{4}I & -D_{12} & D_{12}(\hat{A}_{22} - \hat{A}_{12}^T D_{12})^{-1}(\hat{A}_{23} - \hat{A}_{12}^T D_{13}) - D_{13} \\ 0 & I & -(\hat{A}_{22} - \hat{A}_{12}^T D_{12})^{-1}(\hat{A}_{23} - \hat{A}_{12}^T D_{13}) \\ 0 & 0 & I \end{bmatrix}$$

where I is the 3×3 identity matrix, $D_{12} = \hat{A}_{11}^{-1} \hat{A}_{12}$, $D_{13} = \hat{A}_{11}^{-1} \hat{A}_{13}$. It is easy to show that the inverses of the above submatrices indeed exist. Perform the transformation

$$B = G^T A G,$$

then $B = [b_{ij}]$ is a symmetric matrix where

$$\begin{aligned} b_{11} &= 25a^2, & b_{12} &= -2.5a^2 - 2.5b^2 + 6c^2, \\ b_{13} &= -2.5a^2 + 6b^2 - 2.5c^2, & b_{14} &= 0.06a^2 - 0.03b^2 - 0.03c^2, \\ b_{15} &= -4.04a^2 - 2.99b^2 + 7.03c^2, & b_{16} &= -4.04a^2 + 7.03b^2 - 2.99c^2, \\ b_{17} &= -6.84b^2 + 6.84c^2, & b_{18} &= -0.51a^2 - 0.55b^2 + 1.06c^2, \\ b_{19} &= 0.51a^2 - 1.06b^2 + 0.55c^2, & b_{22} &= 25b^2, \\ b_{23} &= 6a^2 - 2.5b^2 - 2.5c^2, & b_{24} &= 0.80a^2 + 0.17b^2 - 0.97c^2, \\ b_{25} &= -9.13a^2 - 0.03b^2 + 9.16c^2, & b_{26} &= -6.07a^2 + 3.87b^2 + 2.20c^2, \\ b_{27} &= -1.89a^2 - 3.06b^2 + 4.95c^2, & b_{28} &= -1.84a^2 - 0.28b^2 + 2.12c^2, \\ b_{29} &= 1.74a^2 - 1.14b^2 - 0.60c^2, & b_{33} &= 25c^2, \\ b_{34} &= 0.80a^2 - 0.97b^2 + 0.17c^2, & b_{35} &= -6.07a^2 + 2.20b^2 + 3.87c^2, \\ b_{36} &= -9.13a^2 + 9.16b^2 - 0.03c^2, & b_{37} &= 1.89a^2 - 4.95b^2 + 3.06c^2, \\ b_{38} &= -1.7a^2 + 0.60b^2 + 1.14c^2, & b_{39} &= 1.84a^2 - 2.12b^2 + 0.28c^2, \\ b_{44} &= 10.93a^2 + 0.01b^2 + 0.01c^2, & b_{45} &= -0.85a^2 - 0.12b^2 - 0.74c^2, \end{aligned}$$

$$\begin{aligned}
b_{46} &= -0.85a^2 - 0.74b^2 - 0.12c^2, & b_{47} &= -6.08b^2 + 6.08c^2, \\
b_{48} &= 0.12a^2 + 0.49b^2 - 0.61c^2, & b_{49} &= -0.12a^2 + 0.61b^2 - 0.49c^2, \\
b_{55} &= 15.75a^2 + 2.79b^2 + 3.88c^2, & b_{56} &= 0.72a^2 + 0.88b^2 + 0.88c^2, \\
b_{57} &= -2.48a^2 + 0.90b^2 + 1.58c^2, & b_{58} &= -2.10a^2 - 4.84b^2 + 6.85c^2, \\
b_{59} &= -0.04a^2 + 0.35b^2 - 0.31c^2, & b_{66} &= 15.75a^2 + 3.88b^2 + 2.79c^2, \\
b_{67} &= 2.48a^2 - 1.58b^2 - 0.90c^2, & b_{68} &= 0.04a^2 + 0.31b^2 - 0.35c^2, \\
b_{69} &= 2.01a^2 - 6.85b^2 + 4.84c^2, & b_{77} &= 0.75a^2 + 8.42b^2 + 8.42c^2, \\
b_{78} &= 0.91a^2 + 1.19b^2 - 2.03c^2, & b_{79} &= 0.91a^2 - 2.03b^2 + 1.19c^2, \\
b_{88} &= -2.11a^2 + 8.26b^2 + 3.31c^2, & b_{89} &= 0.53a^2 - 0.36b^2 - 0.36c^2, \\
b_{99} &= -2.11a^2 + 3.31b^2 + 8.36c^2.
\end{aligned}$$

Under our assumptions we have

$$\begin{aligned}
b_{11} - \sum_{j \neq 1} |b_{1j}| &\geq 20a^2 + 22b^2 - 22c^2 \geq 3.5a^2 + 3.5b^2, \\
b_{22} - \sum_{j \neq 2} |b_{2j}| &\geq 14.95a^2 + 28.55b^2 - 23.5c^2 \geq 19.46a^2, \\
b_{33} - \sum_{j \neq 3} |b_{3j}| &\geq -17.95a^2 + 13.5b^2 + 21.45c^2 \geq 3.5a^2 + 13.5b^2, \\
b_{44} - \sum_{j \neq 4} |b_{4j}| &\geq 9.23a^2 + 7.5b^2 - 9.2c^2 \geq 2.33a^2 + 0.6b^2, \\
b_{55} - \sum_{j \neq 5} |b_{5j}| &\geq 31.67a^2 + 13.1b^2 - 26.54c^2 \geq 4.95a^2,
\end{aligned}$$

$$\begin{aligned}
b_{66} - \sum_{j \neq 6} |b_{6j}| &\geq -3.31a^2 + 31.06b^2 - 13.1c^2 \geq 1.02a^2, \\
b_{77} - \sum_{j \neq 7} |b_{7j}| &\geq 0.74a^2 + 35.05b^2 - 18.35c^2 \geq 1.17a^2, \\
b_{88} - \sum_{j \neq 8} |b_{8j}| &\geq 4.39a^2 + 16.16b^2 - 11.35c^2 \geq 0.97a^2, \\
b_{99} - \sum_{j \neq 9} |b_{9j}| &\geq -8.61a^2 + 17.11b^2 - 0.36c^2 \geq 2.34a^2.
\end{aligned}$$

Note $a^2 \geq 2S_Q$. So the Gerschgorin theorem implies that the minimum eigenvalue $\lambda_{\min} \geq 1.94S_Q$.

Case 2. Suppose $c^2 \geq \frac{3}{4}(a^2 + b^2)$. Now we take $a^2 = b^2 = 1$, $c^2 = 2$, and perform the same transformation as in Case 1. Then we can similarly show that the minimum eigenvalue of B $\lambda_{\min} \geq 0.14S_Q$.

Summarizing the above two cases and noticing (3.5.18) verify the existence of a constant $\alpha' > 0$ such that

$$I_K(u_h, \Pi_h^* u_h) \geq \frac{1}{768S_Q} \cdot 0.14S_Q Z^T (G^{-1})^T G^{-1} Z \geq \alpha' Z^T Z. \quad (3.5.19)$$

Finally, combining (3.5.6), (3.5.19) and Lemma 3.5.1 yields (3.5.17). $\square$

3.5.4 Error estimates

Theorem 3.5.2 *Let T_h satisfy the assumption of Theorem 3.5.1, and let u and u_h be the solutions to the variational problem (3.4.2) and the cubic element difference scheme (3.5.1) respectively. Then if $u \in H^4(\Omega)$ we have the following error estimate:*

$$\|u - u_h\|_1 \leq Ch^3 |u|_4. \quad (3.5.20)$$

Proof By (3.4.2), (3.5.1) and the *a priori* estimate (3.5.17) we have

$$\|u - u_h\|_1 \leq \|u - \Pi_h u\|_1 + \frac{1}{\alpha} \sup_{\bar{u}_h \in U_h} \frac{a(u - \Pi_h u, \Pi_h^* \bar{u}_h)}{\|\bar{u}_h\|_1}, \quad (3.5.21)$$

where $\Pi_h u$ is the interpolation of u onto U_h . By the interpolation approximation theorem,

$$\|u - \Pi_h u\|_1 \leq Ch^3 |u|_4. \quad (3.5.22)$$

Next we deal with the second term of the right-hand side of (3.5.21). It follows from (3.5.6) and (3.5.7) that

$$a(u - \Pi_h u, \Pi_h^* \bar{u}_h) = \sum_{K \in T_h} I_K(u - \Pi_h u, \Pi_h^* \bar{u}_h), \quad (3.5.23)$$

$$\begin{aligned} & I_K(u - \Pi_h u, \Pi_h^* \bar{u}_h) \\ = & \sum_{l=i,j,k} \left\{ [\bar{u}_h(P_l) - \bar{u}_h(Q)] \int_{\partial K_{P_l}^* \cap K} \frac{\partial(u - \Pi_h u)}{\partial y} dx - \frac{\partial(u - \Pi_h u)}{\partial x} dy \right. \\ & + \frac{\partial \bar{u}_h(P_l)}{\partial x} \left[\int_{K_{P_l}^* \cap K} \frac{\partial(u - \Pi_h u)}{\partial x} dx dy \right. \\ & + \int_{\partial K_{P_l}^* \cap K} \frac{\partial(u - \Pi_h u)}{\partial y} (x - x_l) dx - \frac{\partial(u - \Pi_h u)}{\partial x} (x - x_l) dy \Big] \\ & + \frac{\partial \bar{u}_h(P_l)}{\partial y} \left[\int_{K_{P_l}^* \cap K} \frac{\partial(u - \Pi_h u)}{\partial y} dx dy \right. \\ & \left. \left. + \int_{\partial K_{P_l}^* \cap K} \frac{\partial(u - \Pi_h u)}{\partial y} (y - y_l) dx - \frac{\partial(u - \Pi_h u)}{\partial x} (y - y_l) dy \right] \right\}. \end{aligned} \quad (3.5.24)$$

By the definition of the discrete norm and (3.5.9) we have

$$|\bar{u}_h(P_l) - \bar{u}_h(Q)| \leq C |\bar{u}_h|_{1,h,K}, \quad (3.5.25)$$

$$\left| \frac{\partial \bar{u}_h(P_l)}{\partial x} \right|, \left| \frac{\partial \bar{u}_h(P_l)}{\partial y} \right| \leq Ch^{-1} |\bar{u}_h|_{1,h,K}. \quad (3.5.26)$$

Write $\phi_1 = \frac{\partial(u - \Pi_h u)}{\partial x}$, $\phi_2 = \frac{\partial(u - \Pi_h u)}{\partial y}$, $K_l = K_{P_l}^* \cap K$ and $L = \partial K_{P_l}^* \cap K$. Then, it is easy to see that

$$\begin{aligned} & \left| \int_{K_{P_l}^* \cap K} \frac{\partial(u - \Pi_h u)}{\partial x} dx dy \right| \\ \leq & h \left(\int_{K_l} \phi_1^2 dx dy \right)^{1/2} \leq h |u - \Pi_h u|_{1,K} \leq Ch^4 |u|_{4,K}. \end{aligned} \quad (3.5.27)$$

Similarly

$$\left| \int_{K_{P_l}^* \cap K} \frac{\partial(u - \Pi_h u)}{\partial y} dx dy \right| \leq Ch^4 |u|_{4,K}. \quad (3.5.28)$$

Also note

$$\left| \int_{\partial K_{P_l}^* \cap K} \frac{\partial(u - \Pi_h u)}{\partial y} dx - \frac{\partial(u - \Pi_h u)}{\partial x} dy \right| \leq h^{1/2} \left[\int_L (\phi_1^2 + \phi_2^2) ds \right]^{1/2}, \quad (3.5.29)$$

$$\begin{aligned} & \left| \int_{\partial K_{P_l}^* \cap K} \frac{\partial(u - \Pi_h u)}{\partial y} (x - x_l) dx - \frac{\partial(u - \Pi_h u)}{\partial x} (x - x_l) dy \right| \\ & \leq h^{3/2} \left[\int_L (\phi_1^2 + \phi_2^2) ds \right]^{1/2}, \end{aligned} \quad (3.5.30)$$

$$\begin{aligned} & \left| \int_{\partial K_{P_l}^* \cap K} \frac{\partial(u - \Pi_h u)}{\partial y} (y - y_l) dx - \frac{\partial(u - \Pi_h u)}{\partial x} (y - y_l) dy \right| \\ & \leq h^{3/2} \left[\int_L (\phi_1^2 + \phi_2^2) ds \right]^{1/2}. \end{aligned} \quad (3.5.31)$$

After the transformation $(x, y) \rightarrow (\lambda_j, \lambda_k)$ we have

$$\int_L \phi_i^2 ds \leq h \int_{\tilde{L}} \hat{\phi}_i^2 d\hat{s}, \quad i = 1, 2. \quad (3.5.32)$$

Use the trace theorem on $\hat{K}_Q^*$ to obtain

$$\begin{aligned} & \left| \int_{\tilde{L}} \hat{\phi}_i^2 d\hat{s} \right| \leq C \|\hat{\phi}_i\|_{1,\hat{K}}^2 \leq C(h^{-1}|\phi_i|_{0,K} + |\phi_i|_{1,K})^2 \\ & \leq C(h^{-1}|u - \Pi_h u|_{1,K} + |u - \Pi_h u|_{2,K})^2 \leq Ch^4 |u|_{4,K}^2, \quad i = 1, 2. \end{aligned} \quad (3.5.33)$$

It follows from (3.5.24)-(3.5.33) that

$$|I_K(u - \Pi_h u, \Pi_h^* \bar{u}_h)| \leq Ch^3 |u|_{4,K} |\bar{u}_h|_{1,h,K}. \quad (3.5.34)$$

This together with (3.5.23) and Lemma 3.5.1 gives

$$|a(u - \Pi_h u, \Pi_h^* \bar{u}_h)| \leq Ch^3 |u|_4 |\bar{u}_h|_1. \quad (3.5.35)$$

Combining (3.5.21), (3.5.22) and (3.5.35) implies (3.5.20). This completes the proof. $\square$

3.6 L^2 and Maximum Norm Estimates

Consider the boundary problem of the second order elliptic equation

$$\begin{cases} -\sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + qu = f, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \quad (3.6.1a)$$

$$(3.6.1b)$$

and its generalized difference scheme: Find $u_h \in U_h$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \quad (3.6.2)$$

Here Ω is a polygonal region; the functions $a_{ij} \in W^{1,\infty}(\Omega)$ ($i, j = 1, 2$) and $q \in L^\infty(\Omega)$ satisfy the elliptic condition; $f \in L^2(\Omega)$; U_h is the linear element space related to a quasi-uniform triangulation T_h ; and V_h is the piecewise constant function space corresponding to the barycenter dual decomposition T_h^* . (See §3.2 for details.) In this section we discuss the error estimates in L^2 and maximum norms.

3.6.1 L^2 estimates

Let us introduce an auxiliary problem: Find $w \in H_0^1(\Omega)$ such that

$$a(v, w) = (g, v), \quad \forall v \in H_0^1(\Omega). \quad (3.6.3)$$

Assume the problem is regular, namely for any $g \in L^2(\Omega)$ there exist a unique solution $w \in H_0^1(\Omega) \cap H^2(\Omega)$ and a constant C such that

$$\|w\|_2 \leq C\|g\|_0. \quad (3.6.4)$$

According to the theory of differential equations, problem (3.6.3) is regular when Ω is convex and the coefficients a_{ij} and the function q are sufficiently smooth.

Theorem 3.6.1 *Let u be the solution to problem (3.6.1), u_h to the generalized difference scheme (3.6.2), and $u \in H_0^1(\Omega) \cap W^{3,p}(\Omega)$ ($p > 1$). Then*

$$\|u - u_h\|_0 \leq Ch^2 \|u\|_{3,p}. \quad (3.6.5)$$

Proof We use (3.6.3) with $g = u - u_h$ to get

$$\|u - u_h\|_0^2 = a(u - u_h, w). \quad (3.6.6)$$

Let $\Pi_h w$ and $\Pi_h^* w$ be the interpolation projections of w onto U_h and V_h respectively. Then, obviously the error $u - u_h$ satisfies

$$a(u - u_h, \Pi_h^* w) = 0. \quad (3.6.7)$$

This implies

$$\|u - u_h\|_0^2 = a(u - u_h, w - \Pi_h w) + a(u - u_h, \Pi_h w) - a(u - u_h, \Pi_h^* w). \quad (3.6.8)$$

Notice that the first two bilinear functionals of the right-hand side of (3.6.8) are in the usual sense while the last one is in the sense of generalized functions. It follows from the boundedness of the bilinear functionals, the H^1 -estimate (3.2.36) and the interpolation estimates (3.2.8) and (3.6.4) that

$$\begin{aligned} & |a(u - u_h, w - \Pi_h w)| \\ & \leq C \|u - u_h\|_1 \|w - \Pi_h w\|_1 \\ & \leq Ch^2 \|u\|_2 \|u - u_h\|_0. \end{aligned} \quad (3.6.9)$$

On the other hand, by Green's formula we have

$$\begin{aligned} & a(u - u_h, \Pi_h w) \\ &= \sum_K \int_K \sum_{i,j=1}^2 a_{ij} \frac{\partial(u - u_h)}{\partial x_j} \frac{\partial \Pi_h w}{\partial x_i} dx + \int_{\Omega} q(u - u_h) \Pi_h w dx \\ &= \sum_K \int_K \sum_{i,j=1}^2 [a_{ij}(x) - a_{ij}(Q)] \frac{\partial(u - u_h)}{\partial x_j} \frac{\partial \Pi_h w}{\partial x_i} dx \\ & \quad - \sum_K \int_K \sum_{i,j=1}^2 a_{ij}(Q) \frac{\partial^2 u}{\partial x_i \partial x_j} \Pi_h w dx \\ & \quad + \sum_K \int_{\partial K} \sum_{i,j=1}^2 a_{ij}(Q) \frac{\partial(u - u_h)}{\partial x_j} \cos \langle n, x_i \rangle \Pi_h w ds \\ & \quad + \int_{\Omega} q(u - u_h) \Pi_h w dx, \end{aligned} \quad (3.6.10)$$

$$\begin{aligned}
& a(u - u_h, \Pi_h^* w) \\
&= - \sum_K \sum_{P \in \dot{K}} \int_{\partial K_P^* \cap K} \sum_{i,j=1}^2 a_{ij} \frac{\partial(u - u_h)}{\partial x_j} \cos \langle n, x_i \rangle \Pi_h^* w ds \\
&\quad + \int_{\Omega} q(u - u_h) \Pi_h^* w dx, \\
&= - \sum_K \sum_{P \in \dot{K}} \int_{\partial K_P^* \cap K} \sum_{i,j=1}^2 [a_{ij}(x) - a_{ij}(Q)] \\
&\quad \cdot \frac{\partial(u - u_h)}{\partial x_j} \cos \langle n, x_i \rangle \Pi_h^* w ds \tag{3.6.11} \\
&\quad - \sum_K \int_K \sum_{i,j=1}^2 a_{ij}(Q) \frac{\partial^2 u}{\partial x_i \partial x_j} \Pi_h^* w dx \\
&\quad + \sum_K \int_{\partial K} \sum_{i,j=1}^2 a_{ij}(Q) \frac{\partial(u - u_h)}{\partial x_j} \cos \langle n, x_i \rangle \Pi_h^* w ds \\
&\quad + \int_{\Omega} q(u - u_h) \Pi_h^* w dx.
\end{aligned}$$

Here Q is the barycenter of K , and $\dot{K} = K \cap \dot{\Omega}_h$. Hence

$$a(u - u_h, \Pi_h w) - a(u - u_h, \Pi_h^* w) = \sum_{i=1}^5 E_i(u - u_h, w), \tag{3.6.12}$$

where

$$\begin{aligned}
E_1(u - u_h, w) &= \sum_K \int_K \sum_{i,j=1}^2 [a_{ij}(x) - a_{ij}(Q)] \frac{\partial(u - u_h)}{\partial x_j} \frac{\partial \Pi_h w}{\partial x_i} dx, \\
E_2(u - u_h, w) &= - \sum_K \sum_{P \in \dot{K}} \int_{\partial K_P^* \cap K} \sum_{i,j=1}^2 [a_{ij}(x) - a_{ij}(Q)] \\
&\quad \cdot \frac{\partial(u - u_h)}{\partial x_j} \cos \langle n, x_i \rangle \Pi_h^* w ds,
\end{aligned}$$

$$\begin{aligned}
E_3(u - u_h, w) &= - \sum_K \int_K \sum_{i,j=1}^2 a_{ij}(Q) \frac{\partial^2 u}{\partial x_i \partial x_j} (\Pi_h w - \Pi_h^* w) dx, \\
E_4(u - u_h, w) &= \sum_K \int_K \sum_{i,j=1}^2 a_{ij}(Q) \frac{\partial(u - u_h)}{\partial x_j} \cos\langle n, x_i \rangle (\Pi_h w - \Pi_h^* w) ds, \\
E_5(u - u_h, w) &= \int_Q q(u - u_h) (\Pi_h w - \Pi_h^* w) dx.
\end{aligned}$$

Noticing (3.2.36) and (3.6.4) we have

$$|E_1(u - u_h, w)| \leq Ch|u - u_h|_1|w|_1 \leq Ch^2\|u\|_2\|u - u_h\|_0. \quad (3.6.13)$$

Using the argument in §3.2.4 (cf. the symbols therein and (3.2.44)) we have

$$\begin{aligned}
&|E_2(u - u_h, w)| \\
&= \left| \sum_K \sum_{l=i,j,k} \int_{M_l Q} \sum_{i,j=1}^2 [a_{ij}(x) - a_{ij}(Q)] \right. \\
&\quad \left. \frac{\partial(u - u_h)}{\partial x_j} \cos\langle n, x_i \rangle ds (w_{l+2} - w_{l+1}) \right| \\
&\leq Ch^2 \sum_K |u|_{2,K} |w|_{1,K} \leq Ch^2\|u\|_2\|u - u_h\|_0.
\end{aligned} \quad (3.6.14)$$

In the case of the barycenter decomposition, we have for any element K

$$\int_K (\Pi_h w - \Pi_h^* w) dx = 0.$$

Write $v_j = \frac{\partial u}{\partial x_j}$ for $j = 1, 2$, then $v_j \in W^{2,p}(\Omega)$. By the above equality and the fact that $\frac{\partial \Pi_h v_j}{\partial x_i}$ is a constant on each element K , we have

$$\begin{aligned}
&E_3(u - u_h, w) \\
&= \sum_K \int_K \sum_{i,j=1}^2 a_{ij}(Q) \left(\frac{\partial v_j}{\partial x_i} - \frac{\partial \Pi_h v_j}{\partial x_i} \right) (\Pi_h w - \Pi_h^* w) dx.
\end{aligned}$$

Hence

$$|E_3(u - u_h, w)| \leq C \sum_{j=1}^2 \|v_j - \Pi_h v_j\|_{1,p} \|\Pi_h w - \Pi_h^* w\|_{0,q},$$

where $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Notice the imbedding relations $W^{2,p}(\Omega) \rightarrow C(\bar{\Omega})$ and $H^2(\Omega) \rightarrow W^{1,q}(\Omega)$, and use the interpolation theorem and (3.6.4), then we have

$$|E_3(u - u_h, w)| \leq Ch^2 \sum_{j=1}^2 \|v_j\|_{2,p} \|w\|_{1,q} \leq Ch^2 \|u\|_{3,p} \|u - u_h\|_0. \quad (3.6.15)$$

Now let us estimate E_4 . A direct calculation shows that along any a line L of an element K we have

$$\int_L (\Pi_h w - \Pi_h^* w) ds = 0.$$

Furthermore, since $\frac{\partial u_h}{\partial x_j} \cos\langle n, x_i \rangle$ is a constant on L ,

$$\sum_K \int_{\partial K} \sum_{l,m=1}^2 a_{lm}(Q) \frac{\partial u_h}{\partial x_m} \cos\langle n, x_l \rangle (\Pi_h w - \Pi_h^* w) ds = 0.$$

So

$$E_4 = \sum_K \int_{\partial K} \sum_{l,m=1}^2 a_{lm}(Q) \frac{\partial u}{\partial x_m} \cos\langle n, x_l \rangle (\Pi_h w - \Pi_h^* w) ds.$$

Because we have zero boundary condition on the outer boundary, we only have to consider the integrals on the inner boundaries. Let $K = K_Q$ (see Fig. 3.2.7), and $K_{Q'}$ is a neighbouring element sharing a common side $\bar{P}_i \bar{P}_j$ with K_Q . Denote by n_Q the unit outer normal direction of K_Q along $\bar{P}_i \bar{P}_j$. Obviously, the two line integrals along $\bar{P}_i \bar{P}_j$ differ in the sign. Note that $\frac{\partial \Pi_h u}{\partial x_m}$ is also a constant on $\bar{P}_i \bar{P}_j$. Therefore, we may write E_4 in the form

$$E_4 = \sum_{K_Q} \sum_{\bar{P}_i \bar{P}_j} \int_{\bar{P}_i \bar{P}_j} \sum_{l,m=1}^2 (a_{lm}(Q) - a_{lm}(Q')) \left(\frac{\partial u}{\partial x_m} - \frac{\partial \Pi_h u}{\partial x_m} \right) \cdot \cos\langle n_Q, x_l \rangle (\Pi_h w - \Pi_h^* w) ds.$$

It follows from the smoothness of a_{lm} and the Cauchy inequality that

$$|E_4| \leq Ch \sum_{K_Q} \sum_{\overline{P_i P_j}} \left(\sum_{m=1}^2 \int_{\overline{P_i P_j}} \left| \frac{\partial u}{\partial x_m} - \frac{\partial \Pi_h u}{\partial x_m} \right|^2 ds \right)^{1/2} \cdot \left(\int_{\overline{P_i P_j}} |\Pi_h w - \Pi_h^* w|^2 ds \right)^{1/2}. \quad (3.6.16)$$

As in the proof to (3.2.43) we can use the interpolation property to show that

$$\begin{aligned} & \int_{\overline{P_i P_j}} |\Pi_h w - \Pi_h^* w|^2 ds \\ & \leq Ch \left(h^{-1} |\Pi_h w - \Pi_h^* w|_{0, K_Q \cap K_{P_i}^*} + h^{-1} |\Pi_h w - \Pi_h^* w|_{0, K_Q \cap K_{P_j}^*} \right. \\ & \quad \left. + |\Pi_h w - \Pi_h^* w|_{1, K_Q \cap K_{P_i}^*} + |\Pi_h w - \Pi_h^* w|_{1, K_Q \cap K_{P_j}^*} \right)^2 \\ & \leq Ch |\Pi_h w|_{1, K_Q}^2 \leq Ch \|w\|_{2, K_Q}^2 \leq Ch \|u - u_h\|_{0, K_Q}^2. \end{aligned} \quad (3.6.17a)$$

Similarly

$$\int_{\overline{P_i P_j}} \left| \frac{\partial u}{\partial x_m} - \frac{\partial \Pi_h u}{\partial x_m} \right|^2 ds \leq Ch |u|_{2, K_Q}^2. \quad (3.6.17b)$$

Now we insert (3.6.17) into (3.6.16) to obtain

$$|E_4| \leq Ch^2 \|u\|_2 \|u - u_h\|_0. \quad (3.6.18)$$

The estimate of E_5 is simple:

$$\begin{aligned} & |E_5(u - u_h, w)| \\ & \leq C \|u - u_h\|_1 \|\Pi_h w - \Pi_h^* w\|_0 \\ & \leq Ch^2 \|u\|_2 \|u - u_h\|_0. \end{aligned} \quad (3.6.19)$$

A combination of (3.6.13, 14, 15, 18, 19) yields

$$|a(u - u_h, \Pi_h w) - a(u - u_h, \Pi_h^* w)| \leq Ch^2 \|u\|_{3,p} \|u - u_h\|_0. \quad (3.6.20)$$

Finally, (3.6.5) results from (3.6.8), (3.6.9) and (3.6.20). This completes the proof. $\square$

3.6.2 A maximum estimate and some remarks

Theorem 3.6.2 *Under the conditions of Theorem 3.6.1 there holds the following error estimate in maximum norm*

$$\|u - u_h\|_{0,\infty} \leq Ch\|u\|_{3,p} \quad (p > 1). \quad (3.6.21)$$

Proof First we note that

$$\|u - u_h\|_{0,\infty} \leq \|u - \Pi_h u\|_{0,\infty} + \|\Pi_h u - u_h\|_{0,\infty}. \quad (3.6.22)$$

For the first term on the right-hand side of (3.6.22) there exists an element K such that

$$\|u - \Pi_h u\|_{0,\infty} = \|u - \Pi_h u\|_{0,\infty,K}. \quad (3.6.23)$$

The Sobolev interpolation approximation theorem gives

$$\|u - \Pi_h u\|_{0,\infty,K} \leq Ch|u|_{2,K} \leq Ch|u|_2. \quad (3.6.24)$$

For the second term, the inverse property of the finite element implies that

$$\|\Pi_h u - u_h\|_{0,\infty} \leq Ch^{-1}\|\Pi_h u - u_h\|_0. \quad (3.6.25)$$

The approximation theorem and the L^2 estimate (3.6.5) result in

$$\|\Pi_h u - u_h\|_0 \leq \|\Pi_h u - u\|_0 + \|u - u_h\|_0 \leq Ch^2\|u\|_{3,p}. \quad (3.6.26)$$

Combining (3.6.22)-(3.6.26) yields (3.6.21). $\square$

Remark 1 In Theorems 3.6.1 and 3.6.2 we obtain the error estimates of precisely the same optimal orders as those of the linear finite element method, but we require higher smoothness of the solutions. The reason behind it may be that we can not obtain the approximation order of the derivatives in the piecewise constant function spaces.

Remark 2 The conclusions of this section also hold for the quasi-uniform rectangular mesh and the corresponding center dual

decomposition. To do this we only have to slightly revise the argument by taking Π_h as the bilinear interpolation operator on the rectangular mesh, and to note that on the rectangle K we have

$$\int_K (\Pi_h w - \Pi_h^* w) dx = 0.$$

Remark 3 When the test function space V_h is the piecewise linear function space, the dual argument of the finite element method is still valid, and thus we can deduce the same L^2 estimates as those of the finite element method. (cf. Theorem 2.5.2.)

3.7 Superconvergences

This section is devoted to the superconvergence of the solution to the generalized difference scheme (3.6.2) approximating the second order elliptic boundary problem (3.6.1). We take the linear element space as the trial function space U_h . The test function space is chosen as the piecewise constant function space related to the circumcenter dual decomposition T_h . Our results in this section are also valid for the case of the barycenter dual decomposition.

3.7.1 Weak estimate of interpolations

Now we derive an interpolation weak estimate of the bilinear form corresponding to the generalized difference scheme.

Theorem 3.7.1 *Suppose T_h is a uniform decomposition, that is, the union of any a pair of adjacent triangular elements forms a parallelogram. Also assume $u \in H_0^1(\Omega) \cap H^3(\Omega)$, $w_h \in U_h$. Then*

$$a(u - \Pi_h u, \Pi_h^* w_h) \leq Ch^2 \|u\|_3 \|w_h\|_1. \quad (3.7.1)$$

Proof Take any a triangular element $K_Q \in T_h$ with vertexes $P(x_{1l}, x_{2l})$ ($l = i, j, k$), circumcenter Q and the midpoints M_l ($l = i, j, k$) (cf. Fig. 3.7.1). Then we have (cf. (3.2.39) and (3.2.40))

$$a(u - \Pi_h u, \Pi_h^* w_h) = E_1(u, w_h) + E_2(u, w_h), \quad (3.7.2)$$

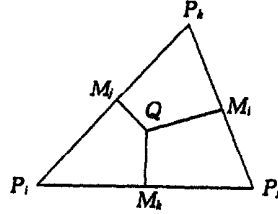


Fig. 3.7.1

where

$$\begin{aligned}
 E_1(u, w_h) &= \sum_K \sum_{l=i,j,k} [w_h(P_{l+2}) - w_h(P_{l+1})] \\
 &\quad \int_{\overline{M_l Q}} \sum_{m,n=1}^2 a_{mn} \frac{\partial(u - \Pi_h u)}{\partial x_n} \cos \langle n_l, x_m \rangle ds,
 \end{aligned} \tag{3.7.3}$$

$$E_2(u, w_h) = \sum_K \sum_{l=i,j,k} w_h(P_l) \int_{K_{P_l}^* \cap K} q(u - \Pi_h u) dx, \tag{3.7.4}$$

where n_l is the outer normal vector along the boundary $\overline{M_l Q}$ of the region $K_{P_{l+1}}^* \cap K$ and we make the convention that $i+1 = j, j+1 = k, k+1 = i$.

Employing the discrete norm (3.2.22) we have

$$|E_2(u, w_h)| \leq C \sum_K \|u - \Pi_h u\|_{0,K} \|w_h\|_{0,h,K} \leq Ch^2 |u|_2 \|w_h\|_0. \tag{3.7.5}$$

In order to estimate $E_1(u, w_h)$, we discuss the two cases where the side $P_j P_k$ belongs to the boundary of Ω or is the common side of two adjacent elements K_Q and $K_{Q'}$, respectively. In the former case, the corresponding terms in (3.7.3) vanish since $w_h|_{\partial\Omega} = 0$. So we concentrate on the latter case. In this case $K_Q \cup K_{Q'}$ is a parallelogram by the uniformity of the decomposition (cf. Fig. 3.7.2). So

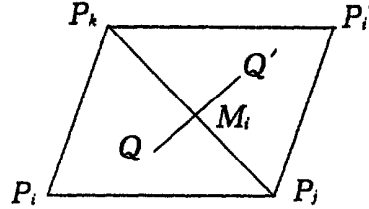


Fig. 3.7.2

we have

$$E_1(u, w_h) = \sum_{\overline{P_j P_k} \not\subset \partial\Omega} [w_h(P_k) - w_h(P_j)] \cdot \int_L \sum_{m,n=1}^2 a_{mn} \frac{\partial(u - \Pi_h u)}{\partial x_n} \cos\langle n_i, x_m \rangle ds, \quad (3.7.6)$$

where $\sum_{\overline{P_j P_k} \not\subset \partial\Omega}$ means the summation over all the sides $\overline{P_j P_k}$ which do not belong to $\partial\Omega$, and $L = Q'M_i \cup M_i Q$. It is easy to see that

$$\begin{aligned} & \left| \int_L a_{mn} \frac{\partial(u - \Pi_h u)}{\partial x_n} \cos\langle n_i, x_m \rangle ds \right| \\ &= \left| a_{mn}(Q) \int_L \frac{\partial(u - \Pi_h u)}{\partial x_n} \cos\langle n_i, x_m \rangle ds \right. \\ & \quad \left. + \int_L [a_{mn}(x) - a_{mn}(Q)] \frac{\partial(u - \Pi_h u)}{\partial x_n} \cos\langle n_i, x_m \rangle ds \right| \quad (3.7.7) \\ &\leq C \sum_{m=1}^2 \left| \int_L \frac{\partial(u - \Pi_h u)}{\partial x_n} dx_m \right| + Ch \int_L \left| \frac{\partial(u - \Pi_h u)}{\partial x_n} \right| ds. \end{aligned}$$

Perform the affine transformation

$$x_m = x_{mi} + (x_{mj} - x_{mi})\lambda_1 + (x_{mk} - x_{mi})\lambda_2, \quad m = 1, 2.$$

Then the parallelogram $K_Q \cup K_{Q'}$ becomes a square $I = \{(\lambda_1, \lambda_2) : 0 \leq \lambda_1 \leq 1, 0 \leq \lambda_2 \leq 1\}$. Suppose the points P_i, M_i and Q are mapped onto $\hat{P}_i, \hat{M}_i$ and $\hat{Q}$ respectively; the segment L is mapped onto $\hat{L}$ and the function $u - \Pi_h u$ onto $\hat{u} - \hat{\Pi}_h \hat{u}$. Then (cf. (3.2.6) and

(3.2.7))

$$\begin{aligned}
& \left| \int_L \frac{\partial(u - \Pi_h u)}{\partial x_n} dx_m \right| \\
&= \left| \int_{\hat{L}} \frac{1}{2S_K} \left[(x_{mk} - x_{mi}) \frac{\partial(\hat{u} - \hat{\Pi}_h \hat{u})}{\partial \lambda_1} + (x_{mi} - x_{mj}) \frac{\partial(\hat{u} - \hat{\Pi}_h \hat{u})}{\partial \lambda_2} \right] \right. \\
&\quad \left. \cdot [(x_{mj} - x_{mi}) d\lambda_1 + (x_{mk} - x_{mi}) d\lambda_2] \right| \\
&\leq C \sum_{m,n=1}^2 \left| \int_{\hat{L}} \frac{\partial(\hat{u} - \hat{\Pi}_h \hat{u})}{\partial \lambda_n} d\lambda_m \right|,
\end{aligned} \tag{3.7.8}$$

where S_K is the area of K . Write

$$J(\hat{u}) = \int_{\hat{L}} \frac{\partial(\hat{u} - \hat{\Pi}_h \hat{u})}{\partial \lambda_n} d\lambda_m. \tag{3.7.9}$$

For any $P \in \mathcal{P}_2(I)$, we note that $\frac{\partial P}{\partial \lambda_n}$ is linear and $\frac{\partial \hat{\Pi}_h P}{\partial \lambda_n}$ is a constant in $\hat{K}_Q$ as well as in $\hat{K}'_Q$. Thus

$$\begin{aligned}
\int_{\hat{L}} \frac{\partial P}{\partial \lambda_n} d\lambda_m &= \frac{\partial P(\frac{1}{2}, \frac{1}{2})}{\partial \lambda_n} L_m, \\
\int_{\hat{L}} \frac{\partial \hat{\Pi}_h P}{\partial \lambda_n} d\lambda_m &= \frac{\partial \hat{\Pi}_h P}{\partial \lambda_n} \Big|_{\hat{K}_Q} \cdot \frac{L_m}{2} + \frac{\partial \hat{\Pi}_h P}{\partial \lambda_n} \Big|_{\hat{K}'_Q} \cdot \frac{L_m}{2} \\
&= [P(1, 0) - P(0, 0) + P(1, 1) - P(0, 1)] L_m / 2 \\
&= \frac{\partial P(\frac{1}{2}, \frac{1}{2})}{\partial \lambda_n} L_m,
\end{aligned}$$

where $L_m = \lambda_m(\hat{Q}') - \lambda_m(\hat{Q})$. Hence

$$J(P) = 0, \quad \forall P \in \mathcal{P}_2(I). \tag{3.7.10}$$

This together with the trace theorem gives

$$|J(\hat{u})| = |J(\hat{u} - P)| \leq \|\hat{u} - P\|_{3,r}.$$

By the quotient space norm theorem and the relationship of the Sobolev semi-norms before and after the affine transformation we have

$$|J(\hat{u})| \leq C \inf_{P \in \mathcal{P}_2(I)} \|\hat{u} - P\|_{3,I} \leq C \|\hat{u}\|_{3,I} \leq Ch^2 |u|_{3,K_Q \cup K_{Q'}}. \quad (3.7.11)$$

It follows from (3.7.8), (3.7.9) and (3.7.11) that

$$\left| \int_L \frac{\partial(u - \Pi_h u)}{\partial x_n} dx_m \right| \leq Ch^2 |u|_{3,K_Q \cup K_{Q'}}. \quad (3.7.12)$$

On the other hand, set $\phi = \frac{\partial(u - \Pi_h u)}{\partial x_n}$, $L_1 = M_i Q$ and $\hat{\phi}(\lambda_1, \lambda_2) = \phi(x_1, x_2)$, then obviously

$$\int_{L_1} \left| \frac{\partial(u - \Pi_h u)}{\partial x_n} \right| ds \leq Ch \int_{L_1} |\hat{\phi}| d\hat{s} \leq Ch \left(\int_{L_1} |\hat{\phi}|^2 d\hat{s} \right)^{1/2}. \quad (3.7.13)$$

Using the trace theorem on $\hat{K}_{P_j}^* \cap \hat{K}$ implies the existence of a constant C independent of K such that (cf. §3.2.4)

$$\left(\int_{L_1} |\hat{\phi}|^2 d\hat{s} \right)^{1/2} \leq C \|\hat{\phi}\|_{1,\hat{K}}.$$

Note

$$|\hat{\phi}|_{0,\hat{K}} \leq Ch^{-1} |\phi|_{0,K}, \quad |\hat{\phi}|_{1,\hat{K}} \leq C |\phi|_{1,K}.$$

Hence

$$\int_{L_1} \left| \frac{\partial(u - \Pi_h u)}{\partial x_n} \right| ds \leq Ch(h^{-1} |\phi|_{0,K} + |\phi|_{1,K}) \leq Ch |u|_{2,K}. \quad (3.7.14)$$

It follows from (3.7.7), (3.7.12) and (3.7.14) that

$$\left| \int_L a_{mn} \frac{\partial(u - \Pi_h u)}{\partial x_n} \cos \langle n_l, x_m \rangle ds \right| \leq Ch^2 \|u\|_{3,K_Q \cup K_{Q'}}. \quad (3.7.15)$$

Noting the linearity of w_h on K_Q we have

$$\begin{aligned} & |w_h(P_k) - w_h(P_j)| \\ &= \left| [x_1(P_k) - x_1(P_j)] \frac{\partial w_h}{\partial x_1} + [x_2(P_k) - x_2(P_j)] \frac{\partial w_h}{\partial x_2} \right| \\ &\leq h \left(\left| \frac{\partial w_h}{\partial x_1} \right| + \left| \frac{\partial w_h}{\partial x_2} \right| \right) \leq C |w_h|_{1,K}. \end{aligned} \quad (3.7.16)$$

By (3.7.6), (3.7.15) and (3.7.16) we have

$$|E_1(u, w_h)| \leq Ch^2 \|u\|_3 \|w_h\|_1. \quad (3.7.17)$$

Finally (3.7.1) is obtained by (3.7.2), (3.7.5) and (3.7.17). $\square$

Next, we try to relax the restriction on the decomposition. We say that a quasi-uniform triangulation T_h is a C -uniform decomposition if for each pair of adjacent nodes P and Q , and the side PP' ($P' \neq Q$) with P as its endpoint, there exists a side QQ' of another triangular element with Q as its endpoint such that $PP'QQ'$ forms a quasi-parallelogram, namely there exists a constant C independent of h such that

$$||\overline{PP'}| - |\overline{QQ'}|| \leq Ch^2.$$

Theorem 3.7.2 Assume T_h is a C -uniform decomposition and $u \in H_0^1(\Omega) \cap H^3(\Omega) \cap W^{2,\infty}(\Omega)$, $w_h \in U_h$. Then

$$|a(u - \Pi_h u, \Pi_h^* w_h)| \leq Ch^2 (\|u\|_3 + \|u\|_{2,\infty}) \|w_h\|_1. \quad (3.7.18)$$

Proof Since T_h is a C -uniform decomposition, the union of any two adjacent element K_Q and $K_{Q'}$ is a quasi-parallelogram. Modify $K_{Q'}$ to get $K_{Q''}$ such that $K_Q \cap K_{Q''}$ is a parallelogram. Then $|Q'Q''|$, the distance between the circumcenters of $K_{Q'}$ and $K_{Q''}$ respectively, is $O(h^2)$. By Theorem 3.7.1, it only remains to estimate the following

$$\begin{aligned} E_3(u, w_h) = & \sum_{P_j P_k \notin \partial\Omega} [w_h(P_k) - w_h(P_j)] \\ & \cdot \int_{Q'Q''} \sum_{m,n=1}^2 a_{mn} \frac{\partial(u - \Pi_h u)}{\partial x_n} \cos\langle n_l, x_m \rangle ds. \end{aligned} \quad (3.7.19)$$

It is easy to see that

$$|E_3(u, w_h)| \leq C \sum_{P_j P_k \notin \partial\Omega} h^3 |u|_{2,\infty} |w_h|_{1,K} \leq Ch^2 |u|_{2,\infty} |w_h|_1. \quad (3.7.20)$$

This implies the desired result and completes the proof. $\square$

The next theorem extends the weak estimation from the case of C -uniform decomposition to the case of piecewise C -uniform decomposition.

Theorem 3.7.3 *Let T_h be a piecewise C -uniform decomposition, where the pieces are divided by several line segments which connect some of the vertexes of Ω and do not intersect each other inside the region. Assume $u \in H_0^1(\Omega) \cap W^{3,\infty}(\Omega)$, $w_h \in U_h$. Then*

$$|a(u - \Pi_h u, \Pi_h^* w_h)| \leq Ch^2 \|u\|_{3,\infty} \|w_h\|_1. \quad (3.7.21)$$

Proof By Theorem 3.7.2 it only remains to show that

$$E_4(u, w_h) = \sum' [w_h(P_k) - w_h(P_j)] \cdot \int_{\bar{M}_i \bar{Q}} \sum_{m,n=1}^2 a_{mn} \frac{\partial(u - \Pi_h u)}{\partial x_n} \cos\langle n_i, x_m \rangle ds, \quad (3.7.22)$$

where $\sum'$ denotes the summation for the cases where $\overline{P_j P_k}$ belongs to the line segments dividing the pieces. Since the number of these line segments is finite, we only have to consider the case where $\overline{P_j P_k}$ belongs to a certain segment and $\bar{M}_i \bar{Q}$'s are on the same side of the segment. So we suppose on one side of the segment $P_0 P_N$ there are N elements as in Fig. 3.7.3 and correspondingly

$$\begin{aligned} E_4^{(1)}(u, w_h) &= \sum_{i=1}^N [w_h(P_i) - w_h(P_{i-1})] \\ &\quad \cdot \int_{\bar{M}_i \bar{Q}_i} \sum_{m,n=1}^2 a_{mn} \frac{\partial(u - \Pi_h u)}{\partial x_n} \cos\langle \tau, x_m \rangle ds \\ &= \sum_{i=1}^{N-1} w_h(P_i) \left(\int_{\bar{M}_i \bar{Q}_i} - \int_{\bar{M}_{i+1} \bar{Q}_{i+1}} \right) \\ &\quad \cdot \sum_{m,n=1}^2 a_{mn} \frac{\partial(u - \Pi_h u)}{\partial x_n} \cos\langle \tau, x_m \rangle ds, \end{aligned} \quad (3.7.23)$$

where we have used the boundary values $w_h(P_0) = w_h(P_N) = 0$ and the notation $\tau = \overline{P_{i-1} P_i} / |\overline{P_{i-1} P_i}|$. By the argument in the proof

to Theorem 3.7.2 we can assume without loss of generality that the elements are all equal to each other. So K_{Q_i} will overlap $K_{Q_{i+1}}$ by displacing it along $\overline{P_0 P_N}$:

$$K_{i+1} = K_i + h_1 \tau \quad (h_1 = |\overline{P_{i-1} P_i}|).$$

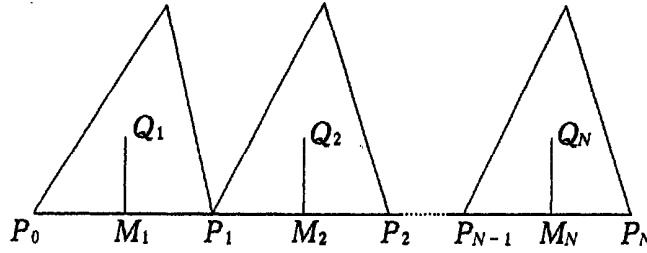


Fig. 3.7.3

Thus

$$\begin{aligned} & \left(\int_{M_i Q_i} - \int_{M_{i+1} Q_{i+1}} \right) a_{mn} \frac{\partial(u - \Pi_h u)}{\partial x_n} \cos \langle \tau, x_m \rangle ds \\ &= - \int_{M_i Q_i} \left[a_{mn}(x + h_1 \tau) \frac{\partial(u - \Pi_h u)}{\partial x_n}(x + h_1 \tau) \right. \\ & \quad \left. - a_{mn}(x) \frac{\partial(u - \Pi_h u)}{\partial x_n}(x) \right] \cos \langle \tau, x_m \rangle ds. \end{aligned} \quad (3.7.24)$$

Thus the fact $a_{mn} \in W^{1,\infty}(\Omega)$ implies that

$$\begin{aligned} & \left| \left(\int_{M_i Q_i} - \int_{M_{i+1} Q_{i+1}} \right) a_{mn} \frac{\partial(u - \Pi_h u)}{\partial x_n} \cos \langle \tau, x_m \rangle ds \right| \\ & \leq C \int_{M_i Q_i} \left| \frac{\partial(u - \Pi_h u)}{\partial x_n}(x + h_1 \tau) - \frac{\partial(u - \Pi_h u)}{\partial x_n}(x) \right| ds \\ & \quad + Ch \int_{M_i Q_i} \left| \frac{\partial(u - \Pi_h u)}{\partial x_n}(x) \right| ds \\ & \leq Ch^3 \|u\|_{3,\infty}. \end{aligned} \quad (3.7.25)$$

It results from (3.7.23) and (3.7.25) that

$$|E_4^{(1)}(u - w_h)| \leq Ch^3 \sum_{i=1}^{N-1} |w_h(P_i)| \cdot \|u\|_{3,\infty} \leq Ch^2 \|u\|_{3,\infty} \|w_h\|_0.$$

This leads to (3.7.21) and completes the proof. $\square$

3.7.2 Superconvergence estimates

Theorem 3.7.4 *Let u be the generalized solution to the second order elliptic boundary value problem (3.6.1) and $u_h \in U_h$ the solution to the generalized difference scheme (3.6.2). Then under the assumptions of Theorems 3.7.1, 3.7.2 and 3.7.3, respectively, we have the following estimates*

$$\|u_h - \Pi_h u\|_1 \leq Ch^2 \|u\|_3, \quad (3.7.26)$$

$$\|u_h - \Pi_h u\|_1 \leq Ch^2 (\|u\|_3 + \|u\|_{2,\infty}), \quad (3.7.27)$$

$$\|u_h - \Pi_h u\|_1 \leq Ch^2 \|u\|_{3,\infty}. \quad (3.7.28)$$

So

$$\left(\frac{1}{r} \sum_{P_0 \in M_h} |\bar{\nabla}(u - u_h)(P_0)|^2 \right)^{1/2} = O(h^2). \quad (3.7.29)$$

Here M_h stands for the set of the optimal stress points of the U_h interpolation (cf. §2.6), $\bar{\nabla}$ the average of the gradient over the elements containing P_0 , and r the number of points in M_h . Therefore, the generalized difference method has the same optimal stress points as the finite element method.

Proof The conclusion follows directly from Theorems 3.7.1-3 of this section, and Theorem 2.6.1. $\square$

Bibliography and Comments

The papers [A-30,62] (cf. [B-57]) extend the generalized difference method to the boundary value problems of second order elliptic partial differential equations on planar regions, discuss the generalized difference methods on triangular and quadrilateral meshes respectively, and propose the basic idea of the method. Error estimates are derived for the cases where the trial function spaces are chosen as piecewise linear and bilinear function spaces, and the test function spaces as piecewise constant function spaces respectively (cf. §3.2 and §3.3). These results further support the following opinion: The convergence order of the generalized difference method is determined mainly by the trial function space, while the test function space influences only the coefficients in the error estimate. These papers have drawn people's attention to the generalized difference method for multidimensional problems (cf. the related references in the end of this book).

The paper [A-55] constructs, using the hierarchical meshes, several generalized difference schemes (including a five-point quadratic scheme, a nine-point bilinear scheme, and a nine-point bi-quadratic scheme etc.) for second order elliptic equations. A numerical analysis is carried out there for the nine-point biquadratic scheme. Recently, [B-62] discusses the generalized difference schemes on arbitrary quadrilateral grids and presents the optimal order convergence estimates. The paper [B-58] applies the generalized difference method to a nonlinear elliptic Dirichlet problem and gives an error estimate (cf. §4.5). In paper [B-55], a $W^{1,p}$ -estimate and an L^2 -estimate of the generalized difference method for second order elliptic equations are studied. Some high order element generalized difference methods on triangular meshes are discussed in [A-41] and [B-14], including the schemes based on the Lagrangian quadratic element and the Hermitian cubic element respectively. The same optimal order error estimates as those of the finite element method are obtained (cf. §3.4 and §3.5). The paper [B-15] investigates the superconvergence of the generalized difference method and shows that the linear element generalized difference method has the same optimal stress points as those of the linear finite element method (cf. §3.7). An L^2 -estimate with an

optimal order is proved for the linear element generalized difference method in [A-10] (cf. §3.6). It can be seen from the discussion of this chapter that generalized difference methods enjoy the same H^1 -estimates as those of finite element methods, save that we require a little bit higher smoothness of the solution for the L^2 -estimate to hold for the linear element scheme.

Finite difference and finite element methods are the two most effective and popular numerical methods for partial differential equations. The finite element method has several remarkable advantages such as that its decompositions is flexible to effectively approximate irregular regions; that it is easy to deal with the boundary conditions as well as the intersection of different media; that it may use high order elements to get a better accuracy without involving too many nodes; that the approximate solution may converge to the generalized solution but not necessarily the classical solution; and that one can use the functional analysis and the Sobolev space theory to provide a systematical numerical analysis. On the other hand, the finite difference method also possesses some useful advantages such as that the construction of the schemes is simple, that the discretization near the node is local and intuitive; and that the computational effort is much less for the same accuracy. But the classical difference method does not share these advantages mentioned above, of the finite element method.

A lot of research has been devoted to the reformation of the classical difference method. A popular approach is to start from the integral conservative forms of boundary value problems and to use numerical integrations to construct conservative difference schemes on irregular networks. From various applied or theoretical points of view, these difference schemes are given different names such as the finite difference method on irregular networks, the box integration method, the balance method, the finite control volume method, the finite volume method, the discretization operator method, and the multi-element balancing method (cf. the references in the end of this book). Most of these methods have mechanical or physical backgrounds and reflect the conservation or the balance of the mechanical

system or other physical systems on the element. From the numerical analysis point of view, these methods are basically regarded as the integral interpolation methods. These difference methods constructed through the integration interpolation over irregular networks possess many advantages of the finite element methods and are effective methods for the numerical computation of partial differential equations. But it is still difficult to construct through this approach the difference schemes with high accuracy.

The generalized difference methods proposed in this chapter essentially reform the integration interpolation methods by absorbing more ideas and tricks of the finite element methods. The first step is to write the boundary value problem into a generalized variational form (the generalized integral conservative form). The second step is to choose the finite element spaces as the trial function spaces (the idea of the finite element method) and the common terms of the local Taylor expansion as the test function spaces (the idea of the difference method). By doing so we keep to the utmost the simplicity of the difference method, while we can construct the difference schemes with high convergence order like the finite element methods and we can employ more results and tricks of the finite element methods in the error analysis.

We end this section by some problems for further investigation.

Problem 1. Further investigate the error estimates in L^2 and maximum norms of the generalized difference methods (the high order element schemes).

Problem 2. Further investigate the superconvergence theory of the generalized difference methods, including the superconvergence of the displacement and the optimal stress point theorem for high order element difference schemes.

Problem 3. Apply the extrapolation method to generalized difference methods and build up the corresponding theory.

Problem 4. Establish the general error estimates in Sobolev spaces for higher order element generalized difference schemes.

Chapter 4

FOURTH ORDER AND NONLINEAR ELLIPTIC EQUATIONS

In this chapter we first consider the mixed and the nonconforming generalized difference methods for fourth order elliptic equations, taking a biharmonic equation as an example. Then we discuss the generalized difference method for a class of nonlinear elliptic equations.

4.1 Mixed Generalized Difference Methods Based on Ciarlet-Raviart Variational Principle

Consider the following Dirichlet problem of the biharmonic equation:

$$\begin{cases} \Delta^2 u \equiv \frac{\partial^4 u}{\partial x_1^4} + 2 \frac{\partial^4 u}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 u}{\partial x_2^4} = f, (x_1, x_2) \in \Omega, & (4.1.1a) \\ u = \frac{\partial u}{\partial n} = 0, (x_1, x_2) \in \partial\Omega, & (4.1.1b) \end{cases}$$

where Ω is a convex polygon region on the plane with boundary $\partial\Omega$, n the unit outer normal vector of $\partial\Omega$, and $f \in L^2(\Omega)$. This kind of boundary problem occupies an important position in, e.g., elastic

mechanics and water kinetics. It is well-known that if $f \in H^{-1}(\Omega)$, then problem (4.1.1) is regular, i.e., there exists a unique solution $u \in H_0^2(\Omega) \cap H^3(\Omega)$ and a constant C such that

$$\|u\|_3 \leq C\|f\|_{-1}, \quad \forall f \in H^{-1}(\Omega). \quad (4.1.2)$$

Introduce a new unknown function $v = -\Delta u$ to reduce problem (4.1.1) into a second order equation:

$$\begin{cases} -\Delta u = v, & (x_1, x_2) \in \Omega, & (4.1.3a) \\ -\Delta v = f, & (x_1, x_2) \in \Omega, & (4.1.3b) \\ u = \frac{\partial u}{\partial n} = 0, & (x_1, x_2) \in \partial\Omega. & (4.1.3c) \end{cases}$$

Multiply (4.1.3a) and (4.1.3b) by $\psi \in H^1(\Omega)$ and $\phi \in H_0^1(\Omega)$ respectively, integrate them on Ω , and use Green's formula and the boundary condition (4.1.3c) to obtain a corresponding variational form: Find $(u, v) \in H_0^1(\Omega) \times H^1(\Omega)$ such that

$$\begin{cases} a(u, \psi) = (v, \psi), & \forall \psi \in H^1(\Omega), & (4.1.4a) \\ a(v, \phi) = (f, \phi), & \forall \phi \in H_0^1(\Omega), & (4.1.4b) \end{cases}$$

where

$$\begin{aligned} a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v dx, \\ (f, \phi) &= \int_{\Omega} f \phi dx. \end{aligned}$$

We can use the regularity condition of (4.1.1) to show the equivalence (the Ciarlet-Raviart variational principle [B-18]) of (4.1.1) and (4.1.4), that is, if u is the solution to (4.1.1) and $v = -\Delta u$ then (u, v) is the solution to (4.1.4); and if (u, v) is the solution to (4.1.4) then u is the solution to (4.1.1) and $v = -\Delta u$.

4.1.1 Mixed generalized difference equations

As in §3.2, let T_h be a quasi-uniform grid and T_h^* the corresponding barycenter dual grid. Suppose U_h is the piecewise linear function space with respect to T_h :

$$U_h = \{u_h \in C(\bar{\Omega}) : u_h|_K \in \mathcal{P}_1, \quad \forall K \in T_h\},$$

and $V_h \in L^2(\Omega)$ is the piecewise constant function space corresponding to T_h^* :

$$V_h = \{v_h : v_h \text{ is a constant on the interior of each } K^* \in T_h^*\}.$$

Set

$$U_{0h} = \{u_h \in U_h : u_h(P_0) = 0, \forall P_0 \in \bar{\Omega}_h \cap \partial\Omega\},$$

$$V_{0h} = \{v_h \in V_h : v_h(P_0) = 0, \forall P_0 \in \bar{\Omega}_h \cap \partial\Omega\},$$

where $\bar{\Omega}_h$ stands for the set of the nodes of T_h .

Multiply (4.1.3a) and (4.1.3b) by $\psi_h \in V_h$ and $\phi_h \in V_{0h}$ respectively, and integrate them on Ω to obtain

$$(-\Delta u, \psi_h) = (v, \psi_h),$$

$$(-\Delta v, \phi_h) = (f, \phi_h).$$

Applying Green's formula to each dual element and noting the boundary condition (4.1.2c) yield

$$(-\Delta u, \psi_h) = - \sum_{P_0 \in \bar{\Omega}_h} \int_{K_{P_0}^*} \Delta u \cdot \psi_h dx \quad (4.1.5a)$$

$$= - \sum_{P_0 \in \bar{\Omega}_h} \psi_h(P_0) \int_{\partial K_{P_0}^*} \frac{\partial u}{\partial n} ds = \sum_{K \in T_h} I_K(u, \psi_h),$$

$$(-\Delta v, \phi_h) = - \sum_{P_0 \in \bar{\Omega}_h} \int_{K_{P_0}^*} \Delta v \cdot \phi_h dx \quad (4.1.5b)$$

$$= - \sum_{P_0 \in \bar{\Omega}_h} \phi_h(P_0) \int_{\partial K_{P_0}^*} \frac{\partial v}{\partial n} ds = \sum_{K \in T_h} I_K(v, \phi_h),$$

where $K = \triangle P_i P_j P_k$ (cf. Fig. 4.1.1), and

$$I_K(u, \psi_h) = \sum_{l=i,j,k} [\psi_h(P_{l+2}) - \psi_h(P_{l+1})] \int_{\overline{M_l Q}} \frac{\partial u}{\partial n_l} ds, \quad (4.1.6a)$$

$$I_K(v, \phi_h) = \sum_{l=i,j,k} [\phi_h(P_{l+2}) - \phi_h(P_{l+1})] \int_{\overline{M_l Q}} \frac{\partial v}{\partial n_l} ds, \quad (4.1.6b)$$

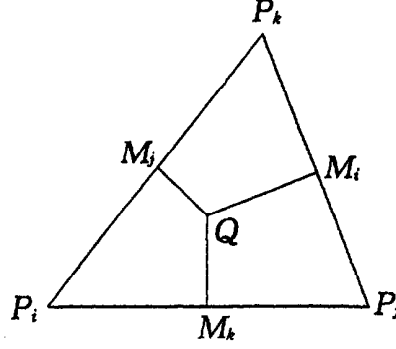


Fig. 4.1.1

where n_i is the unit outer normal direction of $K_{P_{i+1}}^* \cap K$ along $\overline{M_i Q}$, and we make the convention that $i+1 = j, j+1 = k, k+1 = i$.

(4.1.5a) and (4.1.5b) can be respectively rewritten as

$$a(u, \psi_h) = \int_{\Omega} \nabla u \cdot \nabla \psi_h dx,$$

$$a(v, \phi_h) = \int_{\Omega} \nabla v \cdot \nabla \phi_h dx.$$

Here the right-hand sides are in the sense of generalized functions (cf. Chapter 3). Therefore, we have the following variational problem related to (4.1.1): Find $(u, v) \in H_0^1(\Omega) \times H^1(\Omega)$ such that

$$\begin{cases} a(u, \psi) = (v, \psi), & \forall \psi \in S \equiv \bigcup_h V_h, \end{cases} \quad (4.1.7a)$$

$$\begin{cases} a(v, \phi) = (f, \phi), & \forall \phi \in S_0 \equiv \bigcup_h V_{0h}. \end{cases} \quad (4.1.7b)$$

We interpret (4.1.7) as a generalization of the variational form (4.1.1) in the sense of generalized functions. Note that the density of S and S_0 in $L^2(\Omega)$ implies the equivalence of (4.1.7) and (4.1.1).

Based on the above variational form, we define a mixed generalized difference scheme approximating (4.1.1): Find $(u_h, v_h) \in U_{0h} \times U_h$ such that

$$\begin{cases} a(u_h, \psi_h) = (v_h, \psi_h), & \forall \psi_h \in V_h, \\ a(v_h, \phi_h) = (f, \phi_h), & \forall \phi_h \in V_{0h}. \end{cases} \quad (4.1.8a)$$

$$(4.1.8b)$$

Obviously (4.1.8) is a linear system with order $\dim U_{0h} + \dim U_h$.

Let Π_h^* be the interpolation projector from U_h to V_h :

$$\Pi_h^* w_h = \sum_{P_0 \in \bar{\Omega}_h} w_h(P_0) \chi_{P_0}, \quad w_h \in U_h,$$

where χ_{P_0} is the characteristic function of the set $K_{P_0}^*$. Then (4.1.8) is equivalent to

$$\begin{cases} a(u_h, \Pi_h^* \psi_h) = (v_h, \Pi_h^* \psi_h), & \forall \psi_h \in U_h, \\ a(v_h, \Pi_h^* \phi_h) = (f, \Pi_h^* \phi_h), & \forall \phi_h \in U_{0h}. \end{cases}$$

Lemma 4.1.1 *The bilinear form $a(\cdot, \Pi_h^* \cdot)$ is symmetric and positive definite:*

$$a(u_h, \Pi_h^* w_h) = a(w_h, \Pi_h^* u_h), \quad \forall u_h, w_h \in U_h, \quad (4.1.9)$$

$$a(u_h, \Pi_h^* u_h) = |u_h|_1^2, \quad \forall u_h \in U_h. \quad (4.1.10)$$

Proof Since u_h and w_h are piecewise linear functions, $\frac{\partial u_h}{\partial x}$ and $\frac{\partial u_h}{\partial y}$ are respectively constants in each element K . Thus $I_K(\cdot, \cdot)$ can be expressed as (cf. §3.2)

$$\begin{aligned} I_K(u_h, \Pi_h^* w_h) &= \sum_{P \in \bar{K}} w_h(P) \left(\int_{\partial K_P^* \cap K} -\frac{\partial u_h}{\partial x_1} dx_2 + \frac{\partial u_h}{\partial x_2} dx_1 \right) \\ &= \left[\frac{\partial u_h}{\partial x_1} (x_2(M_k) - x_2(M_j)) + \frac{\partial u_h}{\partial x_2} (x_1(M_j) - x_1(M_k)) \right] w_h(P_i) \\ &\quad + \left[\frac{\partial u_h}{\partial x_1} (x_2(M_i) - x_2(M_k)) + \frac{\partial u_h}{\partial x_2} (x_1(M_k) - x_1(M_i)) \right] w_h(P_j) \\ &\quad + \left[\frac{\partial u_h}{\partial x_1} (x_2(M_j) - x_2(M_i)) + \frac{\partial u_h}{\partial x_2} (x_1(M_i) - x_1(M_j)) \right] w_h(P_k) \\ &= \left(\frac{\partial u_h}{\partial x_1} \frac{\partial w_h}{\partial x_1} + \frac{\partial u_h}{\partial x_2} \frac{\partial w_h}{\partial x_2} \right) S_K, \end{aligned} \quad (4.1.11)$$

and

$$a(u_h, \Pi_h^* w_h) = \int_{\Omega} \nabla u_h \cdot \nabla w_h dx. \quad (4.1.12)$$

So (4.1.9) and (4.1.10) hold. This completes the proof. $\square$

Lemma 4.1.2 *There hold the following statements:*

$$(i) (u_h, \Pi_h^* \bar{u}_h) = (\bar{u}_h, \Pi_h^* u_h), \quad \forall u_h, \bar{u}_h \in U_h. \quad (4.1.13)$$

(ii) Write $|||u_h|||_0 = (u_h, \Pi_h^* u_h)^{1/2}$. Then $||| \cdot |||_0$, $\| \cdot \|_{0,h}$ and $\| \cdot \|_0$ are all equivalent on U_h : There exist positive constant c_1 and c_2 independent of U_h such that

$$c_1 \|u_h\|_0 \leq |||u_h|||_0 \leq c_2 \|u_h\|_0, \quad \forall u_h \in U_h. \quad (4.1.14)$$

Proof First we have

$$(u_h, \Pi_h^* \bar{u}_h) = \sum_{K \in T_h} \sum_{l=i,j,k} \bar{u}_h(P_l) \int_{K_{P_l}^* \cap K} u_h dx.$$

By the linearity of u_h we have

$$\begin{aligned} \int_{K_{P_i}^* \cap K} u_h dx &= \frac{1}{3} [u_h(P_i) + u_h(M_j) + u_h(Q)] \cdot \frac{S_K}{6} \\ &\quad + \frac{1}{3} [u_h(P_i) + u_h(M_k) + u_h(Q)] \cdot \frac{S_K}{6} \\ &= \frac{S_K}{108} [22u_h(P_i) + 7u_h(P_j) + 7u_h(P_k)]. \end{aligned}$$

This gives

$$\begin{aligned} &(u_h, \Pi_h^* \bar{u}_h) \\ &= \sum_{K \in T_h} \frac{S_K}{108} [\bar{u}_h(P_i), \bar{u}_h(P_j), \bar{u}_h(P_k)] \begin{bmatrix} 22 & 7 & 7 \\ 7 & 22 & 7 \\ 7 & 7 & 22 \end{bmatrix} \begin{bmatrix} u_h(P_i) \\ u_h(P_j) \\ u_h(P_k) \end{bmatrix}. \end{aligned}$$

This together with Lemma 3.2.1 leads to the desired result and completes the proof. $\square$

Theorem 4.1.1 *The mixed generalized difference scheme (4.1.8) possesses a unique solution.*

Proof We only have to show that the homogeneous equation

$$\begin{cases} a(u_h, \Pi_h^* \psi_h) = (v_h, \Pi_h^* \psi_h), & \forall \psi_h \in U_h \\ a(v_h, \Pi_h^* \phi_h) = 0, & \forall \phi_h \in U_{0h} \end{cases} \quad (4.1.15a) \quad (4.1.15b)$$

admits solely the trivial solution. In fact, if we set $\psi_h = v_h$, $\phi_h = u_h$ in (4.1.15a) and (4.1.15b) respectively, take their subtraction, and use (4.1.9), then we have

$$(v_h, \Pi_h^* v_h) = 0.$$

This together with (4.1.14) implies $v_h = 0$. Thus (4.1.15a) becomes

$$a(u_h, \Pi_h^* \psi_h) = 0, \quad \forall \psi_h \in U_h.$$

So setting $\psi_h = u_h$ and using (4.1.10) yields $u_h = 0$. This completes the proof. $\square$

4.1.2 Error estimates

Let $\Pi_h u$ be the interpolation projection of $u \in H_0^1(\Omega)$ into the linear element space U_{0h} , and $P_h v$ the elliptic projection of $v \in H^1(\Omega)$ into U_h in the following sense:

$$a(P_h v, \Pi_h^* \psi_h) = a(v, \Pi_h^* \psi_h), \quad \forall \psi_h \in U_h, \quad (4.1.16a)$$

$$\int_{\Omega} P_h v dx = \int_{\Omega} v dx. \quad (4.1.16b)$$

For these projections we have

$$|u - \Pi_h u|_m \leq Ch^{2-m}|u|_2, \quad m = 0, 1, \quad (4.1.17)$$

$$\|v - P_h v\|_1 \leq Ch|v|_2. \quad (4.1.18)$$

Here (4.1.18) can be obtained as in §3.2.

Lemma 4.1.3 Suppose that T_h is a quasi-uniform grid and that $u \in H_0^2(\Omega) \cap W^{3,\infty}(\Omega)$, then

$$|a(u - \Pi_h u, \Pi_h^* \psi_h)| \leq Ch^2 \|u\|_{3,\infty} |\psi_h|_1, \quad \forall \psi_h \in U_h. \quad (4.1.19)$$

Proof Note

$$\begin{aligned} & a(u - \Pi_h u, \Pi_h^* \psi_h) \\ &= \sum_{K \in T_h} \sum_{l=i,j,k} [\psi_h(P_{l+2}) - \psi_h(P_{l+1})] \int_{\overline{M_l Q}} \frac{\partial(u - \Pi_h u)}{\partial \tau_l} ds \\ &= \sum' [\psi_h(P_{l+2}) - \psi_h(P_{l+1})] \int_{\overline{M_l Q}} \frac{\partial(u - \Pi_h u)}{\partial \tau_l} ds \\ & \quad + \sum'' [\psi_h(P_{l+2}) - \psi_h(P_{l+1})] \int_{\overline{M_l Q}} \frac{\partial(u - \Pi_h u)}{\partial \tau_l} ds, \end{aligned}$$

where $\sum'$ and $\sum''$ denote the summations for $\overline{P_{l+1}P_{l+2}}$ not belonging to and belonging to the boundary $\partial\Omega$, respectively. In the former case we have (cf. Theorem 3.7.2)

$$\begin{aligned} & \left| \sum' [\psi_h(P_{l+2}) - \psi_h(P_{l+1})] \int_{\overline{M_l Q}} \frac{\partial(u - \Pi_h u)}{\partial \tau_l} ds \right| \\ & \leq Ch^2 (\|u\|_3 + \|u\|_{2,\infty}) |\psi_h|_1. \end{aligned}$$

In the latter case, $\overline{P_{l+1}P_{l+2}}$ belongs to the boundary $\partial\Omega$. Now we expand $\frac{\partial(u - \Pi_h u)}{\partial \tau_l}$ at M_l and use the boundary condition to obtain

$$\left| \frac{\partial(u - \Pi_h u)}{\partial \tau_l} \right| \leq Ch^2 \|u\|_{3,\infty}, \quad x \in \overline{M_l Q}.$$

So we have

$$\left| \sum'' [\psi_h(P_{l+2}) - \psi_h(P_{l+1})] \int_{\overline{M_l Q}} \frac{\partial(u - \Pi_h u)}{\partial \tau_l} ds \right| \leq Ch^2 \|u\|_{3,\infty} |\psi_h|_1. \quad (4.1.22)$$

Finally, (4.1.19) follows from (4.1.20)-(4.1.22). $\square$

Theorem 4.1.2 Let T_h be a quasi-uniform grid, $(u, v) \in H_0^2(\Omega) \times H^2(\Omega)$ the solution to (4.1.9), and $(u_h, v_h) \in U_{0h} \times U_h$ the solution to the mixed generalized difference scheme (4.1.8). Then

$$\|u - u_h\|_1 + \|v - v_h\|_0 \leq Ch(\|u\|_{3,\infty} + |v|_2). \quad (4.1.23)$$

Proof It follows from (4.1.14, 7, 8, 19, 15) that

$$\begin{aligned} \|v_h - P_h v\|_0^2 &\leq Ca(v_h - P_h v, \Pi_h^*(v_h - P_h v)) \\ &= C[a(u_h - \Pi_h u, \Pi_h^*(v_h - P_h v)) + a(\Pi_h u - u, \Pi_h^*(v_h - P_h v)) \\ &\quad + (v - P_h v, \Pi_h^*(v_h - P_h v))] \\ &\leq C[a(v_h - P_h v, \Pi_h^*(u_h - \Pi_h u)) + h\|u\|_{3,\infty}|v_h - P_h v|_1 \\ &\quad + h|v|_2\|v_h - P_h v\|_0]. \end{aligned}$$

It follows from (4.1.7b), (4.1.8b) and (4.1.16a) that

$$a(v_h - P_h v, \Pi_h^*(u_h - \Pi_h u)) = 0.$$

By the above two estimates and the inverse property of the finite element we have

$$\|v_h - P_h v\|_0 \leq Ch(\|u\|_{3,\infty} + |v|_2).$$

This together with (4.1.18) implies

$$\|v - v_h\|_0 \leq Ch(\|u\|_{3,\infty} + |v|_2). \quad (4.1.24)$$

It follows from (4.1.10, 19, 7a, 8a, 18, 23) that

$$\begin{aligned} |\Pi_h u - u_h|_1^2 &= a(\Pi_h u - u, \Pi_h^*(\Pi_h u - u_h)) + a(u - u_h, \Pi_h^*(\Pi_h u - u_h)) \\ &= a(\Pi_h u - u, \Pi_h^*(\Pi_h u - u_h)) + (v - v_h, \Pi_h^*(\Pi_h u - u_h)) \\ &\leq Ch^2\|u\|_{3,\infty}|\Pi_h u - u_h|_1 + Ch(\|u\|_{3,\infty} + |v|_2)\|\Pi_h u - u_h\|_0. \end{aligned}$$

Hence

$$|\Pi_h u - u_h|_1 \leq Ch(\|u\|_{3,\infty} + |v|_2). \quad (4.1.25)$$

A combination of (4.1.24), (4.1.25) and (4.1.17) leads to (4.1.23). This completes the proof. $\square$

4.2 Mixed Generalized Difference Methods Based on Hermann-Miyoshi Variational Principle

Again consider the biharmonic equation (4.1.1). Now let us introduce a new unknown function $v_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ ($i, j = 1, 2$) so as to rewrite (4.1.1) into the following system of second order equations:

$$\begin{cases} \frac{\partial^2 u}{\partial x_i \partial x_j} = v_{ij}, & (x_1, x_2) \in \Omega, i, j = 1, 2, & (4.2.1a) \\ \sum_{i,j=1}^2 \frac{\partial^2 v_{ij}}{\partial x_i \partial x_j} = f, & (x_1, x_2) \in \Omega, & (4.2.1b) \\ u = \frac{\partial u}{\partial n} = 0, & (x_1, x_2) \in \partial\Omega. & (4.2.1c) \end{cases}$$

Write

$$U = H^1(\Omega), \quad U_0 = H_0^1(\Omega),$$

$$\tilde{U} = \{v = (v_{ij}), 1 \leq i, j \leq 2 : v_{12} = v_{21}, v_{ij} \in U\}.$$

Use $\psi \in \tilde{U}$ and $\phi \in U_0$ to multiply (4.2.1a) and (4.2.1b) respectively, integrate them on Ω and employ the Green's formula and the boundary condition (4.2.1c), then we have the variational form corresponding to (4.2.1): Find $(u, v) \in U_0 \times \tilde{U}$ such that

$$a(u, \psi) = -\langle v, \psi \rangle, \quad \forall \psi \in \tilde{U}, \quad (4.2.2a)$$

$$a(\phi, v) = -(f, \phi), \quad \forall \phi \in U_0, \quad (4.2.2b)$$

where

$$a(u, v) = \sum_{i,j=1}^2 \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v_{ij}}{\partial x_j} dx, \quad \forall u \in U, v \in \tilde{U}, \quad (4.2.3)$$

$$\langle v, \psi \rangle = \sum_{i,j=1}^2 \int_{\Omega} v_{ij} \psi_{ij} dx, \quad \forall v, \psi \in \tilde{U}, \quad (4.2.4)$$

$$(f, \phi) = \int_{\Omega} f \phi dx, \quad \forall f, \phi \in U. \quad (4.2.5)$$

We can also use the regularity of (4.1.1) to show the equivalence of (4.2.2) and (4.1.1) [cf. [B-30]: If u is the solution to (4.1.1) and $v = (v_{ij})$ with $v_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ ($i, j = 1, 2$), then (u, v) solves (4.2.2). On the other hand, if (u, v) is the solution to (4.2.2), then u is the solution to (4.1.1) and $v = (v_{ij}) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)$ ($i, j = 1, 2$).

4.2.1 Mixed generalized difference equations

Let U_h, V_h, U_{0h}, V_{0h} be defined as in the last section and

$$\tilde{U}_h = \{u_h = (u_h^{ij}), 1 \leq i, j \leq 2 : u_h^{12} = u_h^{21}, u_h^{ij} \in U_h\},$$

$$\tilde{V}_h = \{v_h = (v_h^{ij}), 1 \leq i, j \leq 2 : v_h^{12} = v_h^{21}, v_h^{ij} \in V_h\}.$$

Based on the variational form (4.2.2), the mixed generalized difference scheme for (4.1.1) is defined as: Find $(u_h, v_h) \in U_{0h} \times \tilde{U}_h$ such that

$$a(u_h, \psi_h) = -\langle v_h, \psi_h \rangle, \quad \forall \psi_h \in \tilde{V}_h, \quad (4.2.6a)$$

$$a(\phi_h, v_h) = -(f, \phi_h), \quad \forall \phi_h \in V_{0h}, \quad (4.2.6b)$$

where $a(\cdot, \cdot)$ is interpreted in the sense of generalized functions (cf. Chapter 3), that is,

$$\begin{aligned} a(u_h, \psi_h) &= \sum_{i,j=1}^2 \int_{\Omega} \frac{\partial u_h}{\partial x_i} \frac{\partial \psi_h^{ij}}{\partial x_j} dx \\ &= \sum_{K_P^* \in T_h^n} \int_{\partial K_P^*} \left(-\frac{\partial u_h}{\partial x_1} \psi_h^{11}(P) dx_2 + \frac{\partial u_h}{\partial x_1} \psi_h^{12}(P) dx_1 \right. \\ &\quad \left. - \frac{\partial u_h}{\partial x_2} \psi_h^{21}(P) dx_2 + \frac{\partial u_h}{\partial x_2} \psi_h^{22}(P) dx_1 \right), \\ &\quad u_h \in U_{0h}, \psi_h \in \tilde{V}_h, \end{aligned} \quad (4.2.7)$$

$$\begin{aligned}
a(\phi_h, v_h) &= \sum_{i,j=1}^2 \int_{\Omega} \frac{\partial \phi_h}{\partial x_i} \frac{\partial v_h^{ij}}{\partial x_j} dx \\
&= \sum_{K_P^* \in T_h^*} \phi_h(P) \int_{\partial K_P^*} -\left(\frac{\partial v_h^{11}}{\partial x_1} + \frac{\partial v_h^{12}}{\partial x_2}\right) dx_2 \\
&\quad + \left(\frac{\partial v_h^{21}}{\partial x_1} + \frac{\partial v_h^{22}}{\partial x_2}\right) dx_1, \\
\phi_h &\in V_{0h}, v_h \in \tilde{U}_h.
\end{aligned} \tag{4.2.8}$$

Scheme (4.2.6) can also be deduced as in §4.1.

If we take ψ_h as the basis function of $\tilde{V}_h$, i.e., for each $P \in \bar{\Omega}_h$, we take ψ_{ij} ($1 \leq i, j \leq 2$) as the characteristic function of K_P^* , then (4.2.6a) becomes

$$\int_{\partial K_P^*} \frac{\partial u_h}{\partial x_1} dx_2 = \int_{K_P^*} v_h^{11} dx, \tag{4.2.9a}$$

$$\int_{\partial K_P^*} \frac{\partial u_h}{\partial x_1} dx_1 = - \int_{K_P^*} v_h^{12} dx, \tag{4.2.9b}$$

$$\int_{\partial K_P^*} \frac{\partial u_h}{\partial x_2} dx_2 = \int_{K_P^*} v_h^{21} dx, \tag{4.2.9c}$$

$$\int_{\partial K_P^*} \frac{\partial u_h}{\partial x_2} dx_1 = - \int_{K_P^*} v_h^{22} dx. \tag{4.2.9d}$$

If ϕ_h is taken as the basis function of V_{0h} , namely, the characteristic function of K_P^* for each $P \in \bar{\Omega}_h$, then (4.2.6b) becomes

$$\int_{\partial K_P^*} -\left(\frac{\partial v_h^{11}}{\partial x_1} + \frac{\partial v_h^{12}}{\partial x_2}\right) dx_2 + \left(\frac{\partial v_h^{21}}{\partial x_1} + \frac{\partial v_h^{22}}{\partial x_2}\right) dx_1 = - \int_{K_P^*} f dx. \tag{4.2.10}$$

Therefore, Scheme (4.2.6) becomes a system of linear algebraic equations (4.2.9) and (4.2.10) with order $(\dim U_{0h} + 3 \dim U_h)$.

4.2.2 Numerical experiments

Consider the numerical solution to the biharmonic boundary value problem:

$$\begin{cases} \Delta^2 u(x, y) = f(x, y), & (x, y) \in \Omega = (0, 1) \times (0, 1), & (4.2.11a) \\ u = \frac{\partial u}{\partial n} = 0, & (x, y) \in \partial\Omega, & (4.2.11b) \end{cases}$$

where

$$f(x, y) = 8[3x^2(1-x)^2 + 3y^2(1-y)^2 + (6x^2 - 6x + 1)(6y^2 - 6y + 1)].$$

The true solution to this problem is

$$u(x, y) = x^2 y^2 (1-x)^2 (1-y)^2.$$

Thirteen point finite difference method

Place a square grid of Ω with a mesh step size $h = 1/n$, the nodes $P_{ij} = (x_i, y_j) = (i/n, j/n)$, $i, j = 0, 1, 2, \dots, n$, and the mesh function of the nodes $u_{ij} = u_h(x_i, y_j)$.

The well-known thirteen point difference scheme is (cf. [c-6]):

$$\begin{aligned} & h^{-4} [20u_{ij} - 8(u_{i,j-1} + u_{i,j+1} + u_{i-1,j} + u_{i+1,j}) \\ & + 2(u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}) \\ & + (u_{i,j-2} + u_{i,j+2} + u_{i-2,j} + u_{i+2,j})] = f(x_i, y_j), \end{aligned} \quad (4.2.12)$$

$$1 \leq i, j \leq n-1.$$

Here when $i = 1, n-1$ or $j = 1, n-1$, we need to define the values on the virtual nodes $P_{ij} = (x_i, y_j)$ (for either $i = -1, n+1$ or $j = -1, n+1$) outside of Ω . For instance, for $P_{-1,j} = (x_{-1}, y_j)$ ($0 \leq j \leq n$) on the left side of Ω , we may employ

$$\frac{u(x_{-1}, y_j) - u(x_1, y_j)}{2h} = \frac{\partial u}{\partial n}(x_0, y_j), \quad 0 \leq j \leq n$$

to define

$$u_{-1,j} = u_{1,j} + 2h \frac{\partial u}{\partial n}(x_0, y_j) = u_{1,j}, \quad 0 \leq j \leq n.$$

Mixed generalized difference methods

On the square mesh with mesh step size $h = 1/n$, we use the diagonals of every small square, parallel to the line $y = x$, to obtain a right angle triangulation together with a corresponding circumcenter dual grid.

Let $u_{ij} = u_h(x_i, y_j)$, $v_{ij}^{kl} = v_h^{kl}(x_i, y_j)$ ($i, j = 0, 1, \dots, n; k, l = 1, 2$) denote the mesh function defined on the nodes $P_{ij} = (x_i, y_j)$ ($i, j = 0, 1, \dots, n$). The generalized difference equation can be easily deduced by a computation of (4.2.9) and (4.2.10). The equation related to the interior point P_{ij} reads:

$$\begin{aligned} & \frac{h^2}{24} (14v_{ij}^{11} + 2v_{i-1,j}^{11} + 2v_{i+1,j}^{11} + 2v_{i,j-1}^{11} \\ & + 2v_{i,j+1}^{11} + v_{i-1,j-1}^{11} + v_{i+1,j+1}^{11}) \\ & = u_{i-1,j} - 2u_{ij} + u_{i+1,j}, \end{aligned} \quad (4.2.9a)'$$

$$\begin{aligned} & \frac{h^2}{24} (14v_{ij}^{12} + 2v_{i-1,j}^{12} + 2v_{i+1,j}^{12} + 2v_{i,j-1}^{12} \\ & + 2v_{i,j+1}^{12} + v_{i-1,j-1}^{12} + v_{i+1,j+1}^{12}) \\ & = \frac{1}{2} (2u_{i,j} + u_{i-1,j-1} + u_{i+1,j+1} \\ & - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}), \end{aligned} \quad (4.2.9b)'$$

$$\begin{aligned} & \frac{h^2}{24} (14v_{ij}^{21} + 2v_{i-1,j}^{21} + 2v_{i+1,j}^{21} + 2v_{i,j-1}^{21} \\ & + 2v_{i,j+1}^{21} + v_{i-1,j-1}^{21} + v_{i+1,j+1}^{21}) \\ & = \frac{1}{2} (2u_{i,j} + u_{i-1,j-1} + u_{i+1,j+1} \\ & - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}), \end{aligned} \quad (4.2.9c)'$$

$$\begin{aligned} & \frac{h^2}{24} (14v_{ij}^{22} + 2v_{i-1,j}^{22} + 2v_{i+1,j}^{22} + 2v_{i,j-1}^{22} \\ & + 2v_{i,j+1}^{22} + v_{i-1,j-1}^{22} + v_{i+1,j+1}^{22}) \\ & = u_{i,j-1} - 2u_{ij} + u_{i,j+1}, \end{aligned} \quad (4.2.9d)'$$

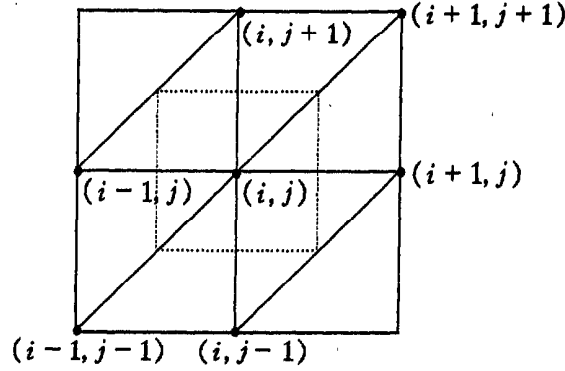


Fig. 4.2.1

$$\begin{aligned}
 & 2v_{ij}^{11} - v_{i-1,j}^{11} - v_{i+1,j}^{11} + 2v_{ij}^{22} - v_{i,j-1}^{22} - v_{i,j+1}^{22} - 2v_{ij}^{12} \\
 & - v_{i-1,j-1}^{12} - v_{i+1,j+1}^{12} + v_{i-1,j}^{12} + v_{i+1,j}^{12} + v_{i,j-1}^{12} + v_{i,j+1}^{12} \\
 & = - \int_{K_{P_{ij}}} f dx \doteq -f(x_i, y_j)h^2.
 \end{aligned} \tag{4.2.10}'$$

(4.2.9b)' and (4.2.9c)' are identical since $v_{ij}^{12} = v_{ij}^{21}$ ($0 \leq i, j \leq n$). So we should take only one of them. To save the space, we do not present here the difference equations with respect to the boundary nodes. We remark that the discrete equations obviously result in a seven point scheme (cf. Fig. 4.2.1).

Numerical results

We use respectively the thirteen point finite difference scheme (FDM) and the mixed generalized difference scheme (GDM) mentioned above to approximate (4.2.11). For $n = 10$, the numerical results and the true solution (TS) are listed in Table 4.2.1 for comparison. This numerical experiment is done in [A-31].

The numerical experiments indicate that the mixed generalized difference method needs less computation time than the corresponding mixed finite element method, while it enjoys a better accuracy as well as a more flexible decomposition than the thirteen point finite difference method.

Table 4.2.1. Numerical Results

(x, y)	FDM u_h	GDM u_h	TS u
(x_1, y_1)	0.00009030	0.00006841	0.00006561
(x_2, y_1)	0.00026062	0.00021000	0.00020736
(x_2, y_2)	0.00075917	0.00065948	0.00065536
(x_3, y_1)	0.00043654	0.00036458	0.00035721
(x_3, y_2)	0.00127511	0.00114762	0.00112896
(x_3, y_3)	0.00214373	0.00198859	0.00194481
(x_4, y_1)	0.00056410	0.00048280	0.00046656
(x_4, y_2)	0.00164913	0.00151576	0.00147456
(x_4, y_3)	0.00277345	0.00261473	0.00254016
(x_4, y_4)	0.00358862	0.00342425	0.00331776
(x_5, y_1)	0.00061032	0.00053178	0.00050625
(x_5, y_2)	0.00178460	0.00166290	0.00160000
(x_5, y_3)	0.00300153	0.00285651	0.00275625
(x_5, y_4)	0.00388386	0.00372713	0.00360000
(x_5, y_5)	0.00420342	0.00404309	0.00390625

4.3 Nonconforming Generalized Difference Method Based on Zienkiewicz Elements

4.3.1 Variational principle

Consider the Dirichlet problem of the biharmonic operator:

$$\begin{cases} \Delta^2 u = f, & (x_1, x_2) \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & (x_1, x_2) \in \partial\Omega, \end{cases} \quad \begin{matrix} (4.3.1a) \\ (4.3.1b) \end{matrix}$$

where Ω is a bounded plane region with a Lipschitz continuous boundary $\partial\Omega$, and $\frac{\partial}{\partial n}$ the derivative operator along the outer normal direction, $f \in L^2(\Omega)$.

As mentioned earlier, the generalized difference method is a kind of difference method based on a variational principle over an irregular network. A basic idea of it is to choose the test function space to be as flexible and simple as possible (usually the piecewise constant or piecewise linear function spaces) so as to reduce the computational effort, while keep the approximation order of the trial function space, ending up with a scheme enjoying both the simplicity of the finite difference method and the accuracy of the finite element method. It is obvious that the usual variational form (where the set of freedoms of the test functions involves second order derivatives) of (4.3.1) requires the test function space to contain piecewise quadratic polynomials, increasing greatly the complexity of the computation. In order to construct a simpler difference scheme we seek another form of the variational principle.

Let σ be a decomposition of Ω , dividing $\bar{\Omega}$ into a sum of finite number of closed subsets K which possess Lipschitz continuous boundaries, have nonempty interiors and share no common inner point:

$$\bar{\Omega} = \bigcup_{K \in \sigma} K,$$

$$\text{int}K_1 \cap \text{int}K_2 = \emptyset, \forall K_1, K_2 \in \sigma, K_1 \neq K_2.$$

Here $\text{int}K$ denotes the interior of K . The family of functions

$$S_\sigma(\Omega) = \{v \in L^2(\Omega) : v|_{\text{int}K} \in \mathcal{P}_1, \forall K \in \sigma\}$$

is called the family of piecewise linear functions related to the decomposition σ , and the family of functions

$$S(\Omega) = \bigcup_{\sigma} S_\sigma(\Omega)$$

the family of piecewise linear functions on Ω .

Take any $v \in S_\sigma(\Omega)$ to multiply (4.3.1a), integrate it on Ω , and use the following Green's formulas on each $K \in \sigma$ (cf. (1.2.10))

$$\int_K \Delta^2 u \cdot v dx = \int_K \Delta u \Delta v dx + \int_{\partial K} \left(\frac{\partial \Delta u}{\partial n} v - \Delta u \frac{\partial v}{\partial n} \right) ds, \quad (4.3.2)$$

$$\begin{aligned}
& \int_K \left(\frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v}{\partial x_1^2} - 2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} \right) dx \\
&= \int_{\partial K} \left(\frac{\partial^2 u}{\partial \tau^2} \frac{\partial v}{\partial n} - \frac{\partial^2 u}{\partial n \partial \tau} \frac{\partial v}{\partial \tau} \right) ds,
\end{aligned} \tag{4.3.3}$$

then we have the following variational form of (4.3.1): Find $u \in H_0^2(\Omega) \cap H^4(\Omega)$ such that

$$a_\sigma(u, v) = (f, v), \quad \forall v \in S_\sigma(\Omega), \sigma \in \{\sigma\}, \tag{4.3.4}$$

where

$$\begin{aligned}
a_\sigma(u, v) &= \int_\Omega v \Delta^2 u dx \\
&= \sum_{K \in \sigma} \int_{\partial K} \left[\left(\frac{\partial \Delta u}{\partial n} v - \Delta u \frac{\partial v}{\partial n} \right) + \mu \left(\frac{\partial^2 u}{\partial \tau^2} \frac{\partial v}{\partial n} - \frac{\partial^2 u}{\partial n \partial \tau} \frac{\partial v}{\partial \tau} \right) \right] ds,
\end{aligned} \tag{4.3.5}$$

where $\frac{\partial}{\partial n}$ and $\frac{\partial}{\partial \tau}$ denote respectively the derivatives along the outer normal and the tangent directions, the value of v on ∂K is interpreted as the continuous extension of the values of $v|_{\text{int}K}$ to ∂K , and μ is a constant. The density of $S(\Omega)$ in $L^2(\Omega)$ implies that u solves (4.3.1) as long as u satisfies (4.3.4). This observation gives the following variational principle.

Theorem 4.3.1 *If $u \in H_0^2(\Omega) \cap H^4(\Omega)$ is a solution to (4.3.1), then u solves (4.3.4), and vice versa.*

4.3.2 Generalized difference schemes based on Zienkiewicz elements

Let Ω be a polygon region, T_h a quasi-uniform triangulation of Ω (h stands for the largest element diameter), and T_h^* a dual grid by connecting the circumcenters of adjacent elements.

Furthermore, for simplicity we assume T_h divide $\bar{\Omega}$ into a sum of finite number of right triangles, and the two right sides of each triangle are parallel to the coordinate axes respectively. Now the dual grid T_h can be regarded as a result of using the perpendicular bisectors parallel to the right sides of the right triangles to divide

$\bar{\Omega}$. Each vertex P of the triangle element (called a node of T_h) is surrounded by a small polygon (in particular, a small rectangle for inner nodes) called a dual element and denoted by K_P^* . (cf. Fig. 4.3.1, where the shaded part is K_P^*).

Choose the trial function space as the finite element space with respect to T_h , the Zienkiewicz triangulation (cf. [p.68, B-17]), of which the function u_h satisfies, at each boundary node P_0 , $u_h(P_0) = \frac{\partial u_h(P_0)}{\partial x_1} = \frac{\partial u_h(P_0)}{\partial x_2} = 0$.

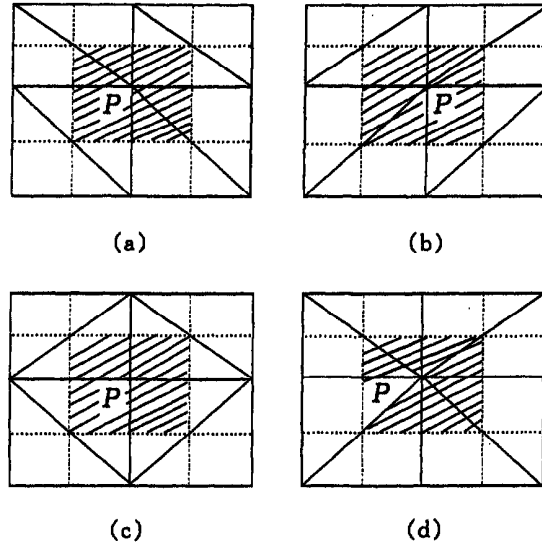


Fig. 4.3.1

The test function space V_h is chosen as the piecewise linear function space corresponding to T_h^* . A function $v_h \in V_h$ satisfies on the boundary nodes $v_h(P_0) = \frac{\partial v_h(P_0)}{\partial x_1} = \frac{\partial v_h(P_0)}{\partial x_2} = 0$. The basis functions with respect to an inner node P_0 are

$$\psi_{P_0}^{(0)}(P) = \begin{cases} 1, & P \in K_{P_0}^*, \\ 0, & P \notin K_{P_0}^*, \end{cases}$$

$$\psi_{P_0}^{(1)}(P) = \begin{cases} x_1 - x_1(P_0), & P \in K_{P_0}^*, \\ 0, & P \notin K_{P_0}^*, \end{cases}$$

$$\psi_{P_0}^{(2)}(P) = \begin{cases} x_2 - x_2(P_0), & P \in K_{P_0}^*, \\ 0, & P \notin K_{P_0}^*. \end{cases}$$

The generalized difference scheme approximating (4.3.1) then becomes: Find $u_h \in U_h$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad (4.3.6)$$

where

$$a_h(u_h, v_h) = \sum_{K \in T_h} I_K(u_h, v_h), \quad (4.3.7)$$

$$I_K(u_h, v_h) = \sum_{P \in \bar{K}} \int_{\partial K_P^* \cap K} \left(\frac{\partial \Delta u_h}{\partial n} v_h - \Delta u_h \frac{\partial v_h}{\partial n} + \frac{\partial^2 u_h}{\partial \tau^2} \frac{\partial v_h}{\partial n} - \frac{\partial^2 u_h}{\partial n \partial \tau} \frac{\partial v_h}{\partial \tau} \right) ds, \quad (4.3.8)$$

where $\bar{K}$ denotes the set of vertexes of the element K .

The trial functions are chosen as Zienkiewicz elements (cf. [p.68, B-17]), so $U_h \not\subset H^2(\Omega)$ and we only have $U_h \in H^1(\Omega)$. The test function V_h is in $L^2(\Omega)$ but not $H^1(\Omega)$. Therefore Scheme (4.3.6) is nonconforming.

The computations of $I_K(u_h, \psi_{P_0}^{(l)})$ ($l = 0, 1, 2$) give the element matrices, and their summation gives the overall matrix, i.e., the coefficient matrix of the linear algebraic system of the approximation problem. Here the computation of the element matrices is simpler than the corresponding nonconforming finite element methods, since the computation here merely involves some line integrals, of which the integral paths are parallel to the coordinate axes; the basis functions $\psi_{P_0}^{(l)}$'s are extremely simple; many terms in I_K are zero; and the nonzero terms are easy to compute.

We do not require the triangulation to satisfy the condition "the three sides of the triangle are parallel to three given directions." Various kinds of grids as illustrated in Fig. 4.3.1 are feasible. The approximation equation related to an inner node P_0 leads to seven point

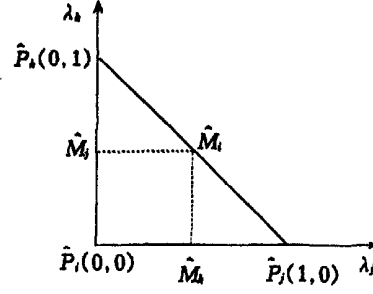


Fig. 4.3.2

generalized difference schemes in the cases of (a) and (b) of Fig. 4.3.1; to a five point scheme in case (c); and to a nine point scheme in case (d).

4.3.3 Error analyses

Take any triangular element $K \in T_h$ with vertexes P_l ($l = i, j, k$). Let P_i denote the right vertex, M_i the midpoint of the side opposite to P_i , S_K the area of K , and $\lambda_K = |\overline{P_i P_j}|^2 / |\overline{P_i P_k}|^2$. Perform an affine transformation

$$\lambda_i = \frac{1}{2S_K} \begin{vmatrix} 1 & x_1 & x_2 \\ 1 & x_1(P_j) & x_2(P_j) \\ 1 & x_1(P_k) & x_2(P_k) \end{vmatrix},$$

$$\lambda_j = \frac{1}{2S_K} \begin{vmatrix} 1 & x_1 & x_2 \\ 1 & x_1(P_k) & x_2(P_k) \\ 1 & x_1(P_i) & x_2(P_i) \end{vmatrix},$$

$$\lambda_k = \frac{1}{2S_K} \begin{vmatrix} 1 & x_1 & x_2 \\ 1 & x_1(P_i) & x_2(P_i) \\ 1 & x_1(P_j) & x_2(P_j) \end{vmatrix}.$$

Then K is mapped onto a reference element $\hat{K}$ on (λ_j, λ_k) plane (Fig. 4.3.2), and accordingly, P_l and M_l are mapped into $\hat{P}_l$ and $\hat{M}_l$ ($l = i, j, k$), respectively. For any $u_h \in U_h$, we have on K that

$$u_h = \sum_{l=i,j,k} (-2\lambda_l^3 + 3\lambda_l^2 + 2\lambda_i\lambda_j\lambda_k)u_h(P_l) + \sum_{l=j,k} \left[(\lambda_l^3 - \lambda_l^2 - \lambda_i\lambda_j\lambda_k) \frac{\partial u_h(P_l)}{\partial \lambda_l} + \sum_{\substack{m=i,j,k \\ m \neq l}} (\lambda_m^2\lambda_l + \frac{1}{2}\lambda_i\lambda_j\lambda_k) \frac{\partial u_h(P_m)}{\partial \lambda_l} \right]. \quad (4.3.9)$$

Define the interpolation operator Π_h^* from U_h to V_h as follows

$$\Pi_h^* w_h = \sum_{P_0 \in \hat{\Omega}_h} \left[w_h(P_0) \psi_{P_0}^{(0)} + \frac{\partial w_h(P_0)}{\partial x_1} \psi_{P_0}^{(1)} + \frac{\partial w_h(P_0)}{\partial x_2} \psi_{P_0}^{(2)} \right], \quad \forall w_h \in U_h, \quad (4.3.10)$$

where $\hat{\Omega}_h$ denotes the set of all the inner nodes of T_h .

A direct computation leads to

$$I_K(u_h, \Pi_h^* w_h) = \frac{1}{16S_K} \delta_K(w_h)^T A_K \delta_K(u_h), \quad (4.3.11)$$

where

$$\delta_K(v) = \left[\frac{\partial v(P_i)}{\partial \lambda_j} + v(P_i) - v(P_j), \frac{\partial v(P_j)}{\partial \lambda_j} + v(P_i) - v(P_j), \right. \\ \left. \frac{\partial v(P_k)}{\partial \lambda_j} + v(P_i) - v(P_j), \frac{\partial v(P_i)}{\partial \lambda_k} + v(P_i) - v(P_k), \right. \\ \left. \frac{\partial v(P_j)}{\partial \lambda_k} + v(P_i) - v(P_k), \frac{\partial v(P_k)}{\partial \lambda_k} + v(P_i) - v(P_k) \right]^T,$$

$$A_K^T = \begin{bmatrix} 6+13\lambda & 2+11\lambda & -4 & 6+6\lambda-\frac{1}{\lambda} & -6\lambda & 6+\frac{1}{\lambda} \\ 4+6\lambda & 4+18\lambda & 0 & 4+6\lambda-\frac{2}{\lambda} & -6\lambda & 4+\frac{2}{\lambda} \\ -2+\lambda & 2-\lambda & 4 & -2-\frac{1}{\lambda} & 0 & -2+\frac{1}{\lambda} \\ 6-\lambda+\frac{6}{\lambda} & 6+\lambda & -\frac{6}{\lambda} & 6+\frac{13}{\lambda} & -4 & 2+\frac{11}{\lambda} \\ -2-\lambda & -2+\lambda & 0 & -2+\frac{1}{\lambda} & 4 & 2-\frac{1}{\lambda} \\ 4-2\lambda+\frac{6}{\lambda} & 4+2\lambda & -\frac{6}{\lambda} & 4+\frac{6}{\lambda} & 0 & 4+\frac{18}{\lambda} \end{bmatrix},$$

where we have written $\lambda = \lambda_K$ for short.

Theorem 4.3.2 *Define*

$$\|u_h\|_h = \left(\sum_{K \in T_h} \frac{1}{S_K} \delta_K(u_h)^T \delta_K(u_h) \right)^{1/2}, \quad \forall u_h \in U_h. \quad (4.3.12)$$

Suppose the triangulation T_h is quasi-uniform: There exists a constant $\lambda_0 > 0$ such that $\lambda_0 \leq \lambda_K \leq \lambda_0^{-1}$. Then $\|\cdot\|_h$ is equivalent to the following $|\cdot|_{2,h}$ norm

$$|u_h|_{2,h} = \left(\sum_{K \in T_h} |u_h|_{2,K}^2 \right)^{1/2}, \quad \forall u_h \in U_h, \quad (4.3.13)$$

which means the existence of positive constants c_1 and c_2 independent of U_h such that

$$c_1 \|u\|_h \leq |u_h|_{2,h} \leq c_2 \|u_h\|_h, \quad \forall u_h \in U_h. \quad (4.3.14)$$

Proof We only have to show the existence of constants $c'_1, c'_2 > 0$ independent of U_h and K such that

$$\frac{c'_1}{S_K} \delta_K(u_h)^T \delta_K(u_h) \leq |u_h|_{2,K}^2 \leq \frac{c'_2}{S_K} \delta_K(u_h)^T \delta_K(u_h), \quad (4.3.15)$$

$$\forall u_h \in U_h, K \in T_h.$$

Express $\frac{\partial^2 u_h}{\partial \lambda_j^2}$, $\frac{\partial^2 u_h}{\partial \lambda_j \partial \lambda_k}$, and $\frac{\partial^2 u_h}{\partial \lambda_k^2}$ as multiplications of vectors and matrices, e.g.,

$$\frac{\partial^2 u_h}{\partial \lambda_j^2} = (\lambda_j, 1, \lambda_k) \begin{bmatrix} 6 & 6 & 0 & 0 & 0 & 0 \\ -4 & -2 & 0 & 0 & 0 & 0 \\ 3 & 2 & -1 & 1 & 1 & 2 \end{bmatrix} \delta_K(u_h).$$

A simple calculation gives

$$\begin{aligned} |u_h|_{2,K}^2 &= \int_K \left[\left(\frac{1}{|\overline{P_i P_j}|^2} \frac{\partial^2 u_h}{\partial \lambda_j^2} \right)^2 + \left(\frac{1}{|\overline{P_i P_j}| \cdot |\overline{P_i P_k}|} \frac{\partial^2 u_h}{\partial \lambda_j \partial \lambda_k} \right)^2 \right. \\ &\quad \left. + \left(\frac{1}{|\overline{P_i P_j}|^2} \frac{\partial^2 u_h}{\partial \lambda_k^2} \right)^2 \right] 2S_K d\lambda_j d\lambda_k \\ &= \frac{1}{S_K} \delta_K(u_h)^T D_K \delta_K(u_h), \end{aligned}$$

where

$$D_K = \frac{1}{24} G^T \text{diag}(\lambda_K D_0, D_0, \lambda_K^{-1} D_0) G,$$

$$G^T = \begin{bmatrix} 6 & -4 & 3 & 3 & -3/2 & 1 & 1 & 0 & 0 \\ 6 & -2 & 2 & 2 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1/2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -3/2 & 3 & 3 & -4 & 6 \\ 0 & 0 & 1 & 1 & 1/2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 2 & -1 & 2 & 2 & -2 & 6 \end{bmatrix},$$

$$D_0 = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 12 & 4 \\ 1 & 4 & 2 \end{bmatrix}.$$

It is easy to verify that D_0 is a positive definite matrix, and that the column vectors of G are linearly independent. Now (4.3.15) holds since $\lambda_0 \leq \lambda_K \leq \lambda_0^{-1}$. This completes the proof. $\square$

Theorem 4.3.3 Suppose the triangulation T_h satisfies $\frac{2}{3} \leq \lambda_K \leq \frac{3}{2}$ ($K \in T_h$), then the bilinear form $a_h(\cdot, \Pi_h^* \cdot)$ is uniformly positive definite, i.e., there is a constant $\alpha > 0$ independent of the subspace U_h such that

$$a_h(u_h, \Pi_h^* u_h) \geq \alpha |u_h|_{2,h}^2, \quad \forall u_h \in U_h. \quad (4.3.16)$$

Proof Replace w_h in (4.3.11) by u_h and write it into a symmetric form:

$$I_K(u_h, \Pi_h^* u_h) = \frac{1}{32S_K} \delta_K(u_h)^T B_K \delta(u_h), \quad (4.3.17)$$

where $B_K = A_K + A_K^T$ is symmetric. We might as well assume $\frac{2}{3} \leq \lambda_K^{-1} \leq 1 \leq \lambda_K \leq \frac{3}{2}$. Let us set $\lambda_K = 1$ in B_K to obtain a matrix $\hat{B}_K$, written in a block form:

$$\hat{B}_K = B_K|_{\lambda_K=1} = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{bmatrix},$$

where each subblock is a 3×3 matrix. Write

$$G_1 = \begin{bmatrix} I & -\hat{B}_{11}^{-1}\hat{B}_{12} \\ 0 & I \end{bmatrix}, \quad G_2 = \text{diag}\left(\frac{2}{3}, \frac{1}{2}, 1, 1, 1, \frac{7}{8}\right),$$

and define the symmetric matrix

$$\tilde{B}_K = G_2^T G_1^T B_K G_1 G_2 = [\tilde{b}_{ij}]_{6 \times 6},$$

where

$$\begin{aligned} \tilde{b}_{11} &= 11.56\lambda + 5.33, & \tilde{b}_{12} &= 5.67\lambda + 2, \\ \tilde{b}_{13} &= 0.67\lambda - 4, & \tilde{b}_{14} &= -4.11\lambda + 0.78 + 3.33\lambda^{-1}, \\ \tilde{b}_{15} &= -0.11\lambda + 0.11, & \tilde{b}_{16} &= -5.25\lambda + 1.17 + 4.08\lambda^{-1}, \\ \tilde{b}_{22} &= 9\lambda + 2, & \tilde{b}_{23} &= -0.5\lambda + 1, \\ \tilde{b}_{24} &= -3.17\lambda + 4.17 - \lambda^{-1}, & \tilde{b}_{25} &= 0.02\lambda - 0.02, \\ \tilde{b}_{26} &= -3.92\lambda + 3.05 + 0.87\lambda^{-1}, & \tilde{b}_{33} &= 8, \\ \tilde{b}_{34} &= -0.17\lambda + 7.17 - 7\lambda^{-1}, & \tilde{b}_{35} &= 0.20\lambda - 0.20, \\ \tilde{b}_{36} &= 0.03\lambda + 4.34 - 4.37\lambda^{-1}, & \tilde{b}_{44} &= 0.13\lambda + 8.93 + 10.22\lambda^{-1}, \\ \tilde{b}_{45} &= 1.65\lambda - 3.37 + 1.11\lambda^{-1}, & \tilde{b}_{46} &= 2.20\lambda + 2.66 + 3.11\lambda^{-1}, \\ \tilde{b}_{55} &= -1.82\lambda + 7.47, & \tilde{b}_{56} &= -0.20\lambda + 3.63, \\ \tilde{b}_{66} &= 2.51\lambda + 4.07 + 19.06\lambda^{-1}, \end{aligned}$$

where we simplify λ_K as λ .

Notice

$$|-0.17\lambda + 7.17 - 7\lambda^{-1}| \leq 0.17\lambda + 6.49 - 6.66\lambda^{-1},$$

$$|1.65\lambda - 3.37 + 1.11\lambda^{-1}| \leq 0.57\lambda - 1.07 + 1.11\lambda^{-1}.$$

Thus, under the given conditions it is easy to check that

$$\begin{aligned}
\tilde{b}_{11} - \sum_{j \neq 1} |\tilde{b}_{1j}| &\geq -2.91\lambda + 1.39 + 7.41\lambda^{-1} \geq 1.96, \\
\tilde{b}_{22} - \sum_{j \neq 2} |\tilde{b}_{2j}| &\geq -2.64\lambda + 6.24 - 0.13\lambda^{-1} \geq 1.23, \\
\tilde{b}_{33} - \sum_{j \neq 3} |\tilde{b}_{3j}| &\geq 0.77\lambda - 7.63 + 11.03\lambda^{-1} \geq 0.75, \\
\tilde{b}_{44} - \sum_{j \neq 4} |\tilde{b}_{4j}| &\geq -10.09\lambda + 5.8 + 14.99\lambda^{-1} \geq 0.65, \\
\tilde{b}_{55} - \sum_{j \neq 5} |\tilde{b}_{5j}| &\geq -2.52\lambda + 5.24 - 1.11\lambda^{-1} \geq 0.72, \\
\tilde{b}_{66} - \sum_{j \neq 6} |\tilde{b}_{6j}| &\geq -8.69\lambda - 2.34 + 25.27\lambda^{-1} \geq 1.46.
\end{aligned}$$

Hence the Gerschgorin theorem guarantees that the minimum eigenvalue of $\tilde{B}_K$ is not less than 0.65. So by (4.3.17) there exists a constant $\bar{\alpha} > 0$ such that

$$\begin{aligned}
&I_K(u_h, \Pi_h^* u_h) \\
&\geq \frac{0.65}{32S_K} \delta_K(u_h)^T (G_2^{-1} G_1^{-1})^T (G_2^{-1} G_1^{-1}) \delta_K(u_h) \\
&\geq \frac{\bar{\alpha}}{S_K} \delta_K(u_h)^T \delta_K(u_h), \quad \forall u_h \in U_h.
\end{aligned}$$

Now the desired result follows from (4.3.7) and Theorem 4.3.2. This completes the proof. $\square$

The existence and the uniqueness of the solution to the generalized difference scheme (4.3.6) results from Theorem 4.3.3.

We pause to point out that a more careful estimation can relax the restriction on the value of λ_K . Our purpose to introduce the last two terms in (4.3.8) is to obtain the uniform ellipticity of the discrete problem. If we multiply these two terms by a parameter μ , then for $\frac{1}{2} \leq \mu \leq \frac{3}{2}$ we have the uniform ellipticity for λ_K in a certain range.

Lemma 4.3.1 *Let Λ_h be a piecewise linear interpolation operator with respect to T_h :*

$$\forall K \in T_h, \Lambda_h w(P_0) = w(P_0) \quad (P_0 \in \dot{K}); \quad \Lambda_h w|_K \in \mathcal{P}_1(K).$$

Then for any $w \in U_h$ and on every $K \in T_h$ we have

$$|\Pi_h^* w_h - \Lambda_h w_h| \leq Ch |w_h|_{2,K}, \quad (4.3.18)$$

$$\left| \frac{\partial}{\partial x_i} (\Pi_h^* w_h - \Lambda_h w_h) \right| \leq C |w_h|_{2,K}, \quad i = 1, 2. \quad (4.3.19)$$

Here C denotes a constant independent of the subspace U_h and the element K .

Proof Let $\overline{P_i P_j}$ and $\overline{P_i P_k}$ be the two sides of the triangle $K = \triangle P_i P_j P_k$, parallel respectively to x_1 and x_2 axes. Then any linear function $\Lambda_h w_h$ on $K_{P_i}^* \cap K$ can be expressed as

$$\begin{aligned} \Lambda_h w_h = & w_h(P_l) + \frac{w_h(P_j) - w_h(P_i)}{x_1(P_j) - x_1(P_i)} (x_1 - x_1(P_l)) \\ & + \frac{w_h(P_k) - w_h(P_i)}{x_2(P_k) - x_2(P_i)} (x_2 - x_2(P_l)). \end{aligned} \quad (4.3.20)$$

Hence it follows from (4.3.10) and (4.3.18) that on $K_{P_i}^* \cap K$

$$\begin{aligned} & \Pi_h^* w_h - \Lambda_h w_h \\ = & \left(\frac{\partial w_h(P_l)}{\partial \lambda_j} + w_h(P_i) - w_h(P_j) \right) \frac{x_1 - x_1(P_l)}{x_1(P_j) - x_1(P_i)} \\ & + \left(\frac{\partial w_h(P_l)}{\partial \lambda_k} + w_h(P_i) - w_h(P_k) \right) \frac{x_2 - x_2(P_l)}{x_2(P_k) - x_2(P_i)}. \end{aligned}$$

This together with (4.3.15) gives

$$\begin{aligned} |\Pi_h^* w_h - \Lambda_h w_h| & \leq 2(\delta_K(w_h)^T \delta_K(w_h))^{1/2} \leq Ch |w_h|_{2,K}, \\ \left| \frac{\partial}{\partial x_1} (\Pi_h^* w_h - \Lambda_h w_h) \right| & \\ \leq & \frac{1}{|x_1(P_j) - x_1(P_i)|} (\delta_K(w_h)^T \delta_K(w_h))^{1/2} \leq C |w_h|_{2,K}. \end{aligned}$$

Similarly we can show (4.3.19) for $i = 2$. This completes the proof. $\square$

In the sequel, we always use u_h to denote the solution to the nonconforming generalized difference scheme (4.3.6).

Theorem 4.3.4 *If the grid T_h satisfies $\frac{2}{3} \leq \lambda_K \leq \frac{3}{2}$ for every $K \in T_h$, and the weak solution $u \in H^3(\Omega) \cap H_0^2(\Omega)$, then we have a constant C independent of the subspace U_h such that*

$$|u - u_h|_{2,h} \leq Ch(|u|_3 + h|f|_0). \quad (4.3.21)$$

Proof Let $\Pi_h u$ be the U_h -interpolation of u , and $w_h = u_h - \Pi_h u$. Then by the uniform ellipticity (4.3.16) and the generalized difference scheme (4.3.6) we have

$$\begin{aligned} \alpha|u_h - \Pi_h u|_{2,h}^2 &\leq a_h(u_h - \Pi_h u, \Pi_h^*(u_h - \Pi_h u)) \\ &= (f, \Pi_h^* w_h - \Lambda_h w_h) + (f, \Lambda_h w_h) - a_h(\Pi_h u, \Pi_h^* w_h). \end{aligned} \quad (4.3.22)$$

Note $\Delta^2 u \in H^{-1}(\Omega)$ since $u \in H^3(\Omega)$. In terms of Green's formula we have for any $v \in C_0^\infty(\Omega)$

$$(f, v) = (\Delta^2 u, v) = - \int_{\Omega} \nabla \Delta u \cdot \nabla v dx.$$

Due to the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$, the above equality is also valid for $v = \Lambda_h w_h \in H_0^1(\Omega)$:

$$(f, \Lambda_h w_h) = - \int_{\Omega} \nabla \Delta u \cdot \nabla \Lambda_h w_h dx. \quad (4.3.23)$$

Use Green's formula on each $K^* \in T_h^*$ and note $\Pi_h^* w_h|_{K^*} \in \mathcal{P}_1(K^*)$ to obtain

$$\begin{aligned} &\sum_{K^* \in T_h^*} \int_{K^*} \nabla \Delta u \cdot \nabla \Pi_h^* w_h dx \\ &= \sum_{K^* \in T_h^*} \left(- \int_{K^*} \Delta u \cdot \Delta \Pi_h^* w_h dx + \int_{\partial K^*} \Delta u \frac{\partial \Pi_h^* w_h}{\partial n} ds \right) \\ &= \sum_{K^* \in T_h^*} \int_{\partial K^*} \Delta u \frac{\partial \Pi_h^* w_h}{\partial n} ds. \end{aligned} \quad (4.3.24)$$

By (4.3.7) and (4.3.8) we have

$$\begin{aligned}
 & a_h(\Pi_h u, \Pi_h^* w_h) \\
 &= \sum_{K \in T_h} \sum_{P \in \dot{K}} \int_{\partial K_P^* \cap K} \left(\frac{\partial \Delta \Pi_h u}{\partial n} \Pi_h^* w_h - \Delta \Pi_h u \frac{\partial \Pi_h^* w_h}{\partial n} \right. \\
 & \quad \left. + \frac{\partial^2 \Pi_h u}{\partial \tau^2} \frac{\partial \Pi_h^* w_h}{\partial n} - \frac{\partial^2 \Pi_h u}{\partial n \partial \tau} \frac{\partial \Pi_h^* w_h}{\partial \tau} \right) ds. \tag{4.3.25}
 \end{aligned}$$

Apply Green's formula (4.3.3) on each $K^* \cap K (\neq \emptyset)$, then we have

$$\begin{aligned}
 & \int_{\partial(K^* \cap K)} \left(-\frac{\partial^2 u}{\partial \tau^2} \frac{\partial v}{\partial n} + \frac{\partial^2 u}{\partial n \partial \tau} \frac{\partial v}{\partial \tau} \right) ds \\
 &= \int_{K^* \cap K} \left(2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v}{\partial x_1^2} \right) dx.
 \end{aligned}$$

The right-hand side vanishes if we take $v = \Pi_h^* w_h$. So

$$\begin{aligned}
 & \sum_{K^* \in T_h^*} \int_{\partial K^*} \left(\frac{\partial^2 u}{\partial \tau^2} \frac{\partial \Pi_h^* w_h}{\partial n} - \frac{\partial^2 u}{\partial n \partial \tau} \frac{\partial \Pi_h^* w_h}{\partial \tau} \right) ds \\
 &= \sum_{K^* \in T_h^*} \sum_{K \cap K^* \neq \emptyset} \int_{\partial K \cap K^*} \left(-\frac{\partial^2 u}{\partial \tau^2} \frac{\partial \Pi_h^* w_h}{\partial n} + \frac{\partial^2 u}{\partial n \partial \tau} \frac{\partial \Pi_h^* w_h}{\partial \tau} \right) ds = 0. \tag{4.3.26}
 \end{aligned}$$

The last equality holds since here the contribution of a line integral is always null no matter whether ∂K is a common side of two adjacent elements or it is on $\partial \Omega$, thanks to the continuity of the integral functions and the boundary condition.

A combination of (4.3.22)-(4.3.26) leads to

$$\begin{aligned}
 & \alpha |u_h - \Pi_h u|_{2,h}^2 \\
 & \leq (f, \Pi_h^* w_h - \Lambda_h w_h) + \sum_{i=1}^4 \sum_{K \in T_h} E_i^K(u, w_h), \tag{4.3.27}
 \end{aligned}$$

where

$$E_1^K(u, w_h) = \sum_{P \in \dot{K}} \int_{K_P^* \cap K} \nabla \Delta u \cdot \nabla (\Pi_h^* w_h - \Lambda_h w_h) dx,$$

$$\begin{aligned}
E_2^K(u, w_h) &= \sum_{P \in \dot{K}} \int_{\partial K_P^* \cap K} -\frac{\partial \Delta \Pi_h u}{\partial n} \Pi_h^* w_h ds, \\
E_3^K(u, w_h) &= \sum_{P \in \dot{K}} \int_{\partial K_P^* \cap K} \left[-\Delta(u - \Pi_h u) + \frac{\partial^2(u - \Pi_h u)}{\partial \tau^2} \right] \frac{\partial \Pi_h^* w_h}{\partial n} ds, \\
E_4^K(u, w_h) &= \sum_{P \in \dot{K}} \int_{\partial K_P^* \cap K} -\frac{\partial^2(u - \Pi_h u)}{\partial n \partial \tau} \frac{\partial \Pi_h^* w_h}{\partial \tau} ds.
\end{aligned}$$

Now we estimate in turn these terms. It follows from the Cauchy inequality, Lemma 4.3.1, (4.3.18) and (4.3.19) that

$$\begin{aligned}
& |(f, \Pi_h^* w_h - \Lambda_h w_h)| \\
& \leq |f|_0 \left(\sum_{K \in T_h} \int_K |\Pi_h^* w_h - \Lambda_h w_h|^2 ds \right)^{1/2} \\
& \leq Ch^2 |f|_0 |w_h|_{2,h},
\end{aligned} \tag{4.3.28}$$

$$|E_1^K(u, w_h)| \leq Ch |u|_{3,K} |w_h|_{2,K}. \tag{4.3.29}$$

It is apparent that

$$E_2^K(u, w_h) = \sum_{P \in \dot{K}} \int_{\partial K_P^* \cap K} -\frac{\partial \Delta \Pi_h u}{\partial n} (\Pi_h^* w_h - \Lambda_h w_h) ds.$$

Write $\phi = \frac{\partial \Delta \Pi_h u}{\partial n}$ and $L_P = \partial K_P^* \cap K$, and note (4.3.18), then we have

$$\begin{aligned}
& |E_2^K(u, w_h)| \\
& \leq \sum_{P \in \dot{K}} \left(\int_{L_P} |\phi|^2 ds \right)^{1/2} \left(\int_{L_P} |\Pi_h^* w_h - \Lambda_h w_h|^2 ds \right)^{1/2} \\
& \leq Ch^{3/2} |w_h|_{2,K} \sum_{P \in \dot{K}} \left(\int_{L_P} |\phi|^2 ds \right)^{1/2}.
\end{aligned} \tag{4.3.30}$$

By the inequality (3.2.43) we have

$$\begin{aligned}
\left(\int_{L_P} |\phi|^2 ds \right)^{1/2} & \leq Ch^{1/2} (h^{-1} |\phi|_{0,K} + |\phi|_{1,K}) \\
& \leq Ch^{-1/2} |\Pi_h u|_{3,K} \leq Ch^{-1/2} |u|_{3,K}.
\end{aligned} \tag{4.3.31}$$

It follows from (4.3.30) and (4.3.31) that

$$|E_2^K(u, w_h)| \leq Ch|u|_{3,K}|w_h|_{2,K}. \quad (4.3.32)$$

Similarly we note

$$E_3^K(u, w_h) = \sum_{P \in \tilde{K}} \int_{\partial K_P^* \cap K} \left[-\Delta(u - \Pi_h u) + \frac{\partial^2(u - \Pi_h u)}{\partial \tau^2} \right] \cdot \frac{\partial(\Pi_h^* w_h - \Lambda_h w_h)}{\partial n} ds.$$

Set $\phi = -\Delta(u - \Pi_h u) + \frac{\partial^2(u - \Pi_h u)}{\partial \tau^2}$, note (4.3.19) and imitate (4.3.31) to obtain

$$\begin{aligned} & |E_3^K(u, w_h)| \\ & \leq \sum_{P \in \tilde{K}} \left(\int_{L_P} |\phi|^2 ds \right)^{1/2} \left(\int_{L_P} \left| \frac{\partial(\Pi_h^* w_h - \Lambda_h w_h)}{\partial n} \right|^2 ds \right)^{1/2} \\ & \leq Ch^{1/2} (h^{-1}|\phi|_{0,K} + |\phi|_{1,K}) \cdot h^{1/2}|w_h|_{2,K} \\ & \leq Ch(h^{-1}|u - \Pi_h u|_{2,K} + |u - \Pi_h u|_{3,K})|w_h|_{2,K} \\ & \leq Ch|u|_{3,K}|w_h|_{2,K}. \end{aligned} \quad (4.3.33)$$

Similarly one can show that

$$|E_4^K(u, w_h)| \leq Ch|u|_{3,K}|w_h|_{2,K}. \quad (4.3.34)$$

Combining (4.3.27)-(4.3.29) and (4.3.32)-(4.3.34) yields

$$\alpha|u_h - \Pi_h u|_{2,h}^2 \leq Ch(|u|_3 + h|f|_0)|u_h - \Pi_h u|_{2,h},$$

$$|u_h - \Pi_h u|_{2,h} \leq Ch(|u|_3 + h|f|_0).$$

This together with a standard estimate for Zienkiewicz elements

$$|u - \Pi_h u|_{2,h} \leq Ch|u|_3$$

implies the error estimate (4.3.21) and completes the proof. $\square$

4.3.4 Numerical experiment

The above nonconforming generalized difference scheme (4.3.6) is used to approximate the following biharmonic equation:

$$\begin{cases} \Delta^2 u = f, & (x, y) \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & (x, y) \in \partial\Omega, \end{cases} \quad \begin{matrix} (4.3.35a) \\ (4.3.35b) \end{matrix}$$

where $\Omega = (0, \pi) \times (0, \pi)$ and $f = 16 \cos 2x \cos 2y - 4 \cos 2x - 4 \cos 2y$.

Place over Ω a uniform square grid with a side size $h = \pi/16$, and then further decompose it into a right angle triangulation, as illustrated in Fig. 4.3.3. Derive a discrete system of equations from Scheme (4.3.6), and solve it by the Seidel iteration method. Table 4.3.1 provides a comparison of the numerical solution and the true solution $u = \sin^2 x \cdot \sin^2 y$. We observe that the maximum relative error of the function values is about 0.003, and that the first partial derivative is also satisfactorily approximated.

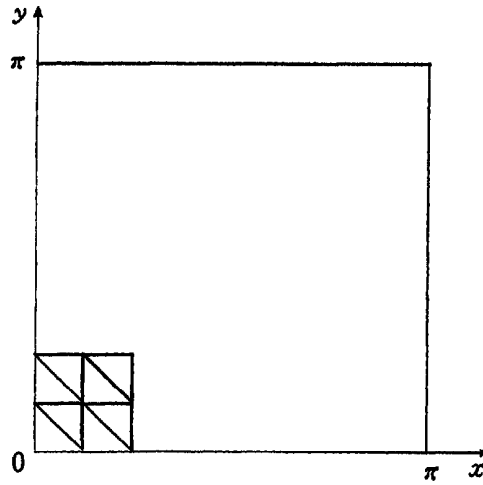


Fig. 4.3.3

Table 4.3.1 Numerical results ($x_i = i\pi/8$, $y_j = j\pi/8$)

(x, y)	u_h	$u_h - u$	$\frac{\partial u_h}{\partial x}$	$\frac{\partial u_h}{\partial x} - \frac{\partial u}{\partial x}$	$\frac{\partial u_h}{\partial y}$	$\frac{\partial u_h}{\partial y} - \frac{\partial u}{\partial y}$
(x_1, y_1)	0.02144	-.00000	0.10642	.00286	.10642	.00286
(x_1, y_2)	0.07328	.00006	0.35783	.00427	.14882	.00238
(x_1, y_3)	0.12495	-.00005	0.60713	.00357	.10481	.00126
(x_1, y_4)	0.14628	-.00017	0.70843	.00132	.00019	.00019
(x_2, y_2)	0.25070	.00070	0.50403	.00403	.50400	.00400
(x_2, y_3)	0.42780	.00102	0.85890	.00535	.35532	.00177
(x_2, y_4)	0.50081	.00081	1.00482	.00482	-.00060	-.00060
(x_3, y_3)	0.73075	.00220	0.60687	.00332	.60677	.00322
(x_3, y_4)	0.85593	.00238	0.71117	.00406	-.00076	-.00076
(x_4, y_4)	1.00308	.00308	-0.00010	-.00010	-.00027	-.00027

4.4 Nonconforming Generalized Difference Methods Based on Adini Elements

4.4.1 Generalized difference scheme

As in the last section, we again consider the Dirichlet problem of the biharmonic equation:

$$\begin{cases} \Delta^2 u = f, & (x_1, x_2) \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & (x_1, x_2) \in \partial\Omega, \end{cases} \quad \begin{matrix} (4.4.1a) \\ (4.4.1b) \end{matrix}$$

Assume that each side of the polygon region $\bar{\Omega}$ is parallel to a coordinate axis. So we can divide $\bar{\Omega}$ to obtain a grid T_h consisting of Adini rectangular elements (cf. [p.364, B-17]). Let h be the maximum diameter of the elements, let the vertexes of the rectangles be the nodes, let $\bar{\Omega}_h$ be the set of all the nodes, and let $\dot{\Omega}_h = \bar{\Omega}_h \setminus \partial\Omega$. The trial function space is chosen as the finite element space with respect to the Adini rectangle, i.e., the incomplete bi-cubic, Hermite type, polynomial space. Any function $u_h \in U_h$ satisfies $u_h(P_0) = \frac{\partial u_h(P_0)}{\partial x_1} = \frac{\partial u_h(P_0)}{\partial x_2} = 0$ at every boundary node P_0 .

In each rectangular element, connect the midpoints of every two opposite sides. Then we re-divide $\bar{\Omega}$ into a sum of some other small rectangles or polygons. Each node P_0 of T_h has a surrounding small rectangle (or possibly a small polygon if P_0 is a boundary node), called a dual element and denoted by $K_{P_0}^*$ (cf. Fig. 4.4.1). The entire dual elements constitute a dual grid T_h^* . The test function space is taken as the piecewise linear function space, which has three basis functions for each interior node P_0 of T_h :

$$\begin{aligned} \psi_{P_0}^{(0)}(P) &= \begin{cases} 1, & P \in K_{P_0}^*, \\ 0, & P \notin K_{P_0}^*, \end{cases} \\ \psi_{P_0}^{(1)}(P) &= \begin{cases} x_1 - x_1(P_0), & P \in K_{P_0}^*, \\ 0, & P \notin K_{P_0}^*, \end{cases} \\ \psi_{P_0}^{(2)}(P) &= \begin{cases} x_2 - x_2(P_0), & P \in K_{P_0}^*, \\ 0, & P \notin K_{P_0}^*. \end{cases} \end{aligned}$$

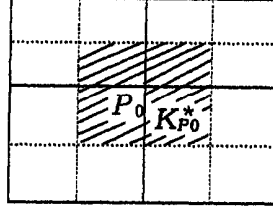


Fig. 4.4.1

Any function $v_h \in V_h$ also satisfies $v_h(P_0) = \frac{\partial v_h(P_0)}{\partial x_1} = \frac{\partial v_h(P_0)}{\partial x_2} = 0$ at any boundary node P_0 .

Based on the variational form (4.3.4), the generalized difference scheme for (4.4.1) is: Find $u_h \in U_h$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad (4.4.2)$$

where

$$a_h(u_h, v_h) = \sum_{K \in T_h} I_K(u_h, v_h), \quad (4.4.3)$$

$$\begin{aligned} & I_K(u_h, v_h) \\ &= \sum_{P \in \dot{K}} \int_{\partial K_P^* \cap K} \left(\frac{\partial \Delta u_h}{\partial n} v_h - \Delta u_h \frac{\partial v_h}{\partial n} + \frac{\partial^2 u_h}{\partial \tau^2} \frac{\partial v_h}{\partial n} - \frac{\partial^2 u_h}{\partial n \partial \tau} \frac{\partial v_h}{\partial \tau} \right) ds, \end{aligned} \quad (4.4.4)$$

where $\dot{K}$ denotes the set of all the vertexes of the element K .

Obviously $U_h \subset H^1(\Omega)$, $V_h \in L^2(\Omega)$. But $U_h \not\subset H^2(\Omega)$, $V_h \not\subset H^1(\Omega)$. Thus (4.4.2) is a nonconforming scheme. We see from the supports of the basis functions $\psi_{P_0}^{(l)}$ ($l = 0, 1, 2$) that the resulting discrete equation is a nine point generalized difference scheme.

We can successively compute the discrete equation

$$a_h(u_h, \psi_{P_0}^{(l)}) = (f, \psi_{P_0}^{(l)}), \quad l = 0, 1, 2, \quad P_0 \in \dot{\Omega}_h$$

for every node $P_0 \in \dot{\Omega}_h$; or we can first compute $I_K(u_h, \psi_{P_0}^{(l)})$ ($l = 0, 1, 2$) to get the element matrices, and then pile them up to obtain

an algebraic system of the discrete problem. Observing that $\psi_{P_0}^{(l)}$ is very simple and that I_K only involves line integrals with the integral paths being parallel to the coordinate axes, the computation here of the element matrices is simpler and more economical compared with the corresponding nonconforming finite element method.

4.4.2 Error estimate

Take any $K \in T_h$ with vertexes P_m ($m = i, j, k, l$), midpoints $M_{ij}, M_{jk}, M_{kl}, M_{li}$ of the sides, the barycenter Q (cf. Fig. 4.4.2(a)), and the area S_K . Set $\Delta x_1 = |\overline{P_i P_j}|$, $\Delta x_2 = |\overline{P_i P_l}|$, and $\lambda_K = (\Delta x_2 / \Delta x_1)^2$. Then the mapping

$$\begin{aligned}\xi &= (x - x_1(P_i)) / \Delta x_1, \\ \eta &= (x_2 - x_2(P_i)) / \Delta x_2\end{aligned}\tag{4.4.5}$$

maps the rectangle K onto a unit square $\hat{K} = [0, 1] \times [0, 1]$, and the nodes $P_i, M_i, Q, \dots$ into $\hat{P}_i, \hat{M}_i, \hat{Q}, \dots$ (cf. Fig. 4.4.2(b).)

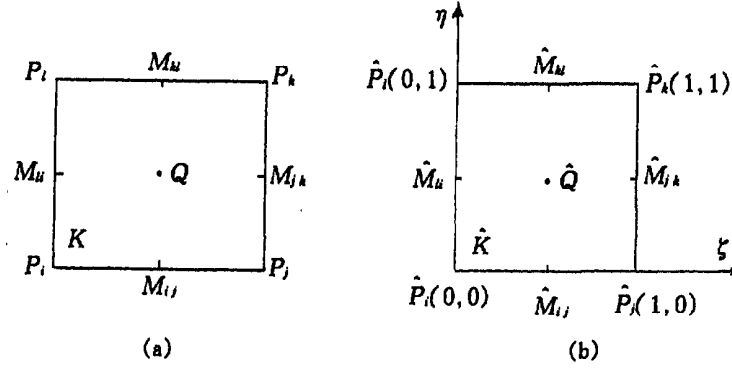


Fig. 4.4.2

Introduce on U_h a discrete norm

$$\|u_h\|_h = \left(\sum_{K \in T_h} \frac{1}{S_K} \delta_K(u_h)^T \delta_K(u_h) \right)^{1/2}, \quad u_h \in U_h, \tag{4.4.6}$$

where

$$\begin{aligned} \delta_K(v) = & \left[v_i - v_j + v_k - v_l, \left(\frac{\partial v}{\partial \xi} \right)_i + v_i - v_j, \left(\frac{\partial v}{\partial \xi} \right)_j + v_i - v_j, \right. \\ & \left(\frac{\partial v}{\partial \xi} \right)_k + v_l - v_k, \left(\frac{\partial v}{\partial \xi} \right)_l + v_l - v_k, \left(\frac{\partial v}{\partial \eta} \right)_i + v_i - v_l, \\ & \left. \left(\frac{\partial v}{\partial \eta} \right)_j + v_j - v_k, \left(\frac{\partial v}{\partial \eta} \right)_k + v_j - v_k, \left(\frac{\partial v}{\partial \eta} \right)_l + v_i - v_l \right]^T. \end{aligned}$$

Here we write for short $v_i = v_h(P_i)$ etc..

Theorem 4.4.1 *Suppose the grid T_h is quasi-uniform: There exists a $\lambda_0 > 0$ such that $\lambda_0 \leq \lambda_K \leq \lambda_0^{-1}$, $\forall K \in T_h$. Then the norm $\|\cdot\|_h$ is equivalent to the norm $|\cdot|_{2,h}$ defined as follows*

$$|u_h|_{2,h} = \left(\sum_{K \in T_h} |u_h|_{2,K}^2 \right)^{1/2}, \quad u_h \in U_h,$$

namely, there exist constants $C_1, C_2 > 0$ independent of U_h such that

$$C_1 \|u_h\|_h \leq |u_h|_{2,h} \leq C_2 \|u_h\|_h, \quad \forall u_h \in U_h. \quad (4.4.7)$$

Proof We only have to show the existence of constants $C'_1, C'_2 > 0$ satisfying

$$\begin{aligned} \frac{C'_1}{S_K} \delta_K(u_h)^T \delta_K(u_h) &\leq |u_h|_{2,K}^2 \leq \frac{C'_2}{S_K} \delta_K(u_h)^T \delta_K(u_h), \\ &\forall u_h \in U_h, \quad K \in T_h. \end{aligned} \quad (4.4.8)$$

The definition of the Adini element on K is as follows:

$$\begin{aligned} u_h = & (-2\xi^3\eta + 2\xi^3 + 3\xi^2\eta - 3\xi^2 - 2\xi\eta^3 \\ & + 3\xi\eta^2 - \xi\eta + 2\eta^3 - 3\eta^2 + 1)(u_h)_i \\ & - (-2\xi^3\eta + 2\xi^3 + 3\xi^2\eta - 3\xi^2 - 2\xi\eta^3 + 3\xi\eta^2 - \xi\eta)(u_h)_j \\ & + (-2\xi^3\eta + 3\xi^2\eta - 2\xi\eta^3 + 3\xi\eta^2 - \xi\eta)(u_h)_k \end{aligned}$$

$$\begin{aligned}
& -(-2\xi^3\eta + 3\xi^2\eta - 2\xi\eta^3 + 3\xi\eta^2 - \xi\eta + 2\eta^3 - 3\eta^2)(u_h)_l \\
& + (-\xi^3\eta + \xi^3 + 2\xi^2\eta - 2\xi^2 - \xi\eta + \xi)\left(\frac{\partial u_h}{\partial \xi}\right)_i \\
& + (-\xi^3\eta + \xi^3 + \xi^2\eta - \xi^2)\left(\frac{\partial u_h}{\partial \xi}\right)_j + (\xi^3\eta - \xi^2\eta)\left(\frac{\partial u_h}{\partial \xi}\right)_k \\
& + (\xi^3\eta - 2\xi^2\eta + \xi\eta)\left(\frac{\partial u_h}{\partial \xi}\right)_l \\
& + (-\xi\eta^3 + 2\xi\eta^2 - \xi\eta + \eta^3 - 2\eta^2 + \eta)\left(\frac{\partial u_h}{\partial \eta}\right)_i \\
& + (\xi\eta^3 - 2\xi\eta^2 + \xi\eta)\left(\frac{\partial u_h}{\partial \eta}\right)_j \\
& + (\xi\eta^3 - \xi\eta^2)\left(\frac{\partial u_h}{\partial \eta}\right)_k + (-\xi\eta^3 + \xi\eta^2 + \eta^3 - \eta^2)\left(\frac{\partial u_h}{\partial \eta}\right)_l.
\end{aligned} \tag{4.4.9}$$

Let us express $\frac{\partial^2 u_h}{\partial x_1^2}$, $\frac{\partial^2 u_h}{\partial x_1 \partial x_2}$ and $\frac{\partial^2 u_h}{\partial x_2^2}$ as multiplications of vectors and matrices:

$$\begin{aligned}
\frac{\partial^2 u_h}{\partial x_1^2} &= \frac{1}{\Delta x_1^2} \frac{\partial^2 u_h}{\partial \xi^2} = \frac{1}{\Delta x_1^2} (\xi\eta, \xi, \eta, 1) G_1 \delta_K(u_h), \\
\frac{\partial^2 u_h}{\partial x_1 \partial x_2} &= \frac{1}{\Delta x_1 \Delta x_2} \frac{\partial^2 u_h}{\partial \xi \partial \eta} = \frac{1}{\Delta x_1 \Delta x_2} (\xi^2, \eta^2, \xi, \eta, 1) G_2 \delta_K(u_h), \\
\frac{\partial^2 u_h}{\partial x_2^2} &= \frac{1}{\Delta x_2^2} \frac{\partial^2 u_h}{\partial \eta^2} = \frac{1}{\Delta x_2^2} (\xi\eta, \xi, \eta, 1) G_3 \delta_K(u_h),
\end{aligned}$$

where

$$\begin{aligned}
G_1 &= \begin{bmatrix} 0 & -6 & -6 & 6 & 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & -2 & -4 & 0 & 0 & 0 & 0 \\ 0 & -4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
G_2 &= \begin{bmatrix} 0 & -3 & -3 & 3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 3 & 3 & -3 \\ 0 & 4 & 2 & -2 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & -4 & -2 & 2 \\ 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \end{bmatrix},
\end{aligned}$$

$$G_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -6 & 6 & 6 & -6 \\ 0 & 0 & 0 & 0 & 0 & 4 & -4 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & -2 \end{bmatrix}.$$

A direct computation leads to

$$\begin{aligned} |u_h|_{2,K}^2 &= \int_K \left[\frac{1}{\Delta x_1^4} \left(\frac{\partial^2 u_h}{\partial \xi^2} \right)^2 + \frac{1}{\Delta x_1^2 \Delta x_2^2} \left(\frac{\partial^2 u_h}{\partial \xi \partial \eta} \right)^2 \right. \\ &\quad \left. + \frac{1}{\Delta x_2^4} \left(\frac{\partial^2 u_h}{\partial \eta^2} \right)^2 \right] \Delta x_1 \Delta x_2 d\xi d\eta \\ &= \frac{1}{S_K} \delta_K(u_h)^T G^T \text{diag}(\lambda_K D_0, D_1, \lambda_K^{-1} D_0) G \delta_K(u_h), \end{aligned} \quad (4.4.10)$$

where

$$\begin{aligned} D_0 &= \int_K (\xi\eta, \xi, \eta, 1)(\xi\eta, \xi, \eta, 1)^T d\xi d\eta, \\ D_1 &= \int_K (\xi^2, \eta^2, \xi, \eta, 1)(\xi^2, \eta^2, \xi, \eta, 1)^T d\xi d\eta, \\ G^T &= [G_1^T G_2^T G_3^T]. \end{aligned}$$

One can easily verify that D_0 and D_1 are positive definite matrices, and that the column vectors of G are linearly independent. (4.4.8) finally follows by use of the quasi-uniform condition of the decomposition. This completes the proof. $\square$

Define an interpolation operator $\Pi_h^* : U_h \rightarrow V_h$ as follows:

$$\Pi_h^* w_h = \sum_{P_0 \in \bar{\Omega}_h} \left[w_h(P_0) \psi_{P_0}^{(0)} + \frac{\partial w_h(P_0)}{\partial x_1} \psi_{P_0}^{(1)} + \frac{\partial w_h(P_0)}{\partial x_2} \psi_{P_0}^{(2)} \right]. \quad (4.4.11)$$

Theorem 4.4.2 Suppose the grid T_h satisfies the condition $\frac{2}{3} \leq \lambda_K \leq \frac{3}{2}$ for all $K \in T_h$. Then the bilinear form $a_h(\cdot, \Pi_h^* \cdot)$ is uniformly positive definite: There exists a constant $\alpha > 0$ independent of U_h such that

$$a_h(u_h, \Pi_h^* u_h) \geq \alpha |u_h|_{2,h}^2, \quad \forall u_h \in U_h. \quad (4.4.12)$$

In the sequel we use u and u_h to denote the weak solution of (4.4.1) and the solution of the nonconforming generalized difference scheme (4.4.2), respectively.

Theorem 4.4.3 *Assume that $\frac{2}{3} \leq \lambda_K \leq \frac{3}{2}$ for all $K \in T_h$, and that the weak solution $u \in H^3(\Omega) \cap H_0^2(\Omega)$, then there exists a constant C independent of U_h such that*

$$|u - u_h|_{2,h} \leq Ch(|u|_3 + h|f|_0). \quad (4.4.13)$$

The proofs to Theorems 4.4.2 and 4.4.3 are omitted to save the space, cf. [A-8].

4.4.3 Numerical example

The nonconforming generalized difference method (4.4.2) is used to approximate the following biharmonic equation:

$$\begin{cases} \Delta^2 u = f, & (x_1, x_2) \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & (x_1, x_2) \in \partial\Omega, \end{cases}$$

where $\Omega = (0, \pi) \times (0, \pi)$ and

$$f(x, y) = 16 \cos 2x \cos 2y - 4 \cos 2x - 4 \cos 2y.$$

Table 4.4.1 Numerical results $(x_i = i\pi/8, y_j = j\pi/8)$

(x, y)	u_h	$u_h - u$	$\frac{\partial u_h}{\partial x}$	$\frac{\partial u_h}{\partial x} - \frac{\partial u}{\partial x}$	$\frac{\partial u_h}{\partial y}$	$\frac{\partial u_h}{\partial y} - \frac{\partial u}{\partial y}$
(x_1, y_1)	0.02135	-0.00010	0.10306	-0.00049	0.10306	-0.00069
(x_1, y_2)	0.07289	-0.00033	0.35232	-0.00123	0.14589	-0.00056
(x_1, y_3)	0.12448	-0.00051	0.60188	0.00168	0.10326	-0.00079
(x_1, y_4)	0.14587	-0.00058	0.70529	-0.00181	-0.00010	-0.00010
(x_2, y_2)	0.24924	-0.00076	0.49923	-0.00077	0.49921	-0.00079
(x_2, y_3)	0.42583	-0.00095	0.85314	-0.00041	0.35334	-0.00022
(x_2, y_4)	0.49901	-0.00099	0.99983	-0.00017	-0.00040	-0.00040
(x_3, y_3)	0.72764	-0.00091	0.60391	0.00036	0.60387	0.00032
(x_3, y_4)	0.85272	-0.00083	0.70774	0.00063	-0.00006	-0.00006
(x_4, y_4)	0.99932	-0.00068	-0.00002	-0.00002	-0.00006	-0.00006

We place a square grid over Ω with $h = \pi/16$, and use (4.4.2). The resulting generalized difference solution is compared with the true solution $u = \sin^2 x \sin^2 y$ in Table 4.4.1. We observe that u_h , $\frac{\partial u_h}{\partial x}$ and $\frac{\partial u_h}{\partial y}$ are all good approximations. We note that this method behaves better than the Ziekiewicz generalized difference method, cf. the numerical example in §4.3.4.

4.5 Second Order Nonlinear Elliptic Equations

In this section we are concerned with the following Dirichlet problem of the second order nonlinear elliptic equation:

$$\begin{cases} -\nabla(a(x, y, u)\nabla u) = f(x, y), & (x, y) \in \Omega, \\ u(x, y) = 0, & (x, y) \in \partial\Omega, \end{cases} \quad \begin{matrix} (4.5.1a) \\ (4.5.1b) \end{matrix}$$

where Ω is a plane bounded region with a sufficiently smooth boundary $\partial\Omega$; $a(x, y, u)$ is a twice continuously differentiable mapping from $\bar{\Omega} \times R$ to $[\alpha_1, \alpha_2]$ ($0 < \alpha_1 < \alpha_2$); and all the second order partial derivatives of $a(x, y, u)$ are bounded on $\bar{\Omega} \times R$. By the Schauder theory and [B-25], if $f \in C^\alpha(\bar{\Omega})$ for some integer $\alpha > 0$, then (4.5.1) has a unique weak solution u , and $u \in C^{2+\alpha}(\bar{\Omega})$. Set

$$\begin{aligned} A(w; u, v) &= (a(x, y, w)\nabla u, \nabla v) \\ &= \int_{\Omega} a(x, y, w)\nabla u \cdot \nabla v dx dy. \end{aligned} \quad (4.5.2)$$

Then, a weak form of (4.5.1) is: Find $u \in H_0^1(\Omega)$ such that

$$A(u; u, v) = (f, v), \quad \forall v \in H_0^1(\Omega). \quad (4.5.3)$$

4.5.1 Generalized difference scheme

Write for short $a(x, y, w) = a(w)$ and suppose Ω is a convex polygonal region. As in §3.2 we place a triangulation T_h and its dual grid T_h^* over $\bar{\Omega}$. Let h be the maximum diameter of all the triangular elements of T_h . Also assume T_h and T_h^* are quasi-uniform, so that there exist

constants $c_1, c_2 > 0$ such that

$$c_1 h^2 \leq S_K \leq c_2 h^2, \quad \forall K \in T_h,$$

$$c_1 h^2 \leq S_{K^*} \leq c_2 h^2, \quad \forall K^* \in T_h^*,$$

where S_K and S_{K^*} stand for the areas of K and K^* respectively.

We choose the trial function space as the standard finite element space with respect to T_h . Corresponding to the freedom

$$l_{P_0}^{i,j} : u \rightarrow \frac{\partial^{i+j} u(P_0)}{\partial x^i \partial y^j}$$

for each interpolation node $P_0 = (x_0, y_0)$ of U_h , we take the i, j term

$$\psi_{P_0}^{(i,j)}(P) = \begin{cases} (x - x_0)^i (y - y_0)^j / (i!j!), & P = (x, y) \in K_{P_0}^* \\ 0, & P \notin K_{P_0}^* \end{cases}$$

of the local Taylor expansion as the basis function of the test function space V_h . In particular, if U_h is a linear element space (piecewise linear polynomial space) with the following freedom related to the node P_0

$$l_{P_0}^{(0,0)} : u \rightarrow u(P_0),$$

then V_h is a piecewise constant function space, of which the basis function for P_0 is a characteristic function of $K_{P_0}^*$:

$$\psi_{P_0}^{(0,0)}(P) = \begin{cases} 1, & P = (x, y) \in K_{P_0}^*, \\ 0, & P \notin K_{P_0}^*. \end{cases}$$

Now the generalized difference scheme for (5.5.1) is: Find $u_h \in U_h$ such that

$$A(u_h; u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad (4.5.4)$$

where

$$\begin{aligned} & A(u_h; u_h, v_h) \\ &= \int_{\Omega} a(u_h) \nabla u_h \cdot \nabla v_h \, dx dy \\ &= \int_{\Omega} a(u_h) \left(\frac{\partial u_h}{\partial x} \frac{\partial v_h}{\partial x} + \frac{\partial u_h}{\partial y} \frac{\partial v_h}{\partial y} \right) dx dy. \end{aligned} \quad (4.5.5)$$

Here $\frac{\partial u_h}{\partial y}$ and $\frac{\partial v_h}{\partial y}$ etc. should be interpreted in the sense of generalized functions. Suppose ψ_h is a basis function of V_h , whose support is a dual element $K_{P_0}^*$, then

$$\begin{aligned} & \int_{\Omega} a(u_h) \nabla u_h \cdot \nabla v_h dx dy \\ &= \int_{K_{P_0}^*} a(u_h) \nabla u_h \cdot \nabla v_h dx dy - \int_{\partial K_{P_0}^*} a(u_h) \frac{\partial u_h}{\partial n} ds, \end{aligned} \quad (4.5.5)'$$

where n is the unit outer normal vector along $\partial K_{P_0}^*$.

In particular when U_h is chosen as the linear element space corresponding to T_h and V_h as the piecewise constant function space, then we have the following linear element difference scheme:

$$A(u_h; u_h, \psi_{P_0}^{(0,0)}) = \int_{K_{P_0}^*} f dx dy, \quad \forall P_0 \in \Omega_h, \quad (4.5.6)$$

where

$$\begin{aligned} & A(u_h; u_h, \psi_{P_0}^{(0,0)}) \\ &= - \int_{K_{P_0}^*} a(u_h) \frac{\partial u_h}{\partial n} ds \\ &= - \int_{\partial K_{P_0}^*} a(u_h) \frac{\partial u_h}{\partial x} dy + \int_{\partial K_{P_0}^*} a(u_h) \frac{\partial u_h}{\partial y} dx. \end{aligned} \quad (4.5.7)$$

Various kinds of numerical methods can be used to compute the line integrals in (4.5.7). For instance, as shown in Fig. 4.5.1, one can write the integral on $\partial K_{P_0}^*$ as a sum of integrals on the fold line segments $\overline{Q_1 M_2 Q_2}, \dots, \overline{Q_6 M_1 Q_1}$, then employ, e.g., the following quadrature formula:

$$\begin{aligned} & - \int_{\overline{Q_1 M_2 Q_2}} a(u_h) \frac{\partial u_h}{\partial n} ds \\ & \doteq - (|\overline{Q_1 M_2}| + |\overline{M_2 Q_2}|) (a(u_h))_{M_2} (u_h(P_2) - u_h(P_0)) / |\overline{P_0 P_2}|, \end{aligned}$$

where $u_h(M_2) = (u_h(P_2) + u_h(P_0))/2$. If T_h^* is a circumcenter dual grid, then (4.5.6) is identical with a finite difference equation derived by an integral conservation law. (See, e.g., [C-6,7].)

Another way is to write the integral on $\partial K_{P_0}^*$ as a sum of integrals on the fold line segments $\overline{M_1 Q_1 M_2}, \dots, \overline{M_6 Q_6 M_1}$, and to use the quadrature formula:

$$\int_{\overline{M_1 Q_1 M_2}} a(u_h) \frac{\partial u_h}{\partial x} dy \doteq (y_{M_2} - y_{M_1})(a(u_h))_{Q_1} \frac{\partial u_h(Q_1)}{\partial x},$$

$$\int_{\overline{M_1 Q_1 M_2}} a(u_h) \frac{\partial u_h}{\partial y} dx \doteq (x_{M_2} - x_{M_1})(a(u_h))_{Q_1} \frac{\partial u_h(Q_1)}{\partial y}.$$

This leads to another sort of difference equation.

Let $\Pi_h^* : U_h \rightarrow V_h$ be an interpolation operator:

$$\Pi_h^* \bar{u}_h = \sum_{P \in \hat{\Omega}_h} \bar{u}(P_0) \psi_{P_0}^{(0,0)}(P), \quad \forall \bar{u}_h \in U_h,$$

then we can rewrite the generalized difference scheme (4.5.4) into an equivalent form:

$$A(u_h; u_h, \Pi_h^* \bar{u}_h) = (f, \Pi_h^* \bar{u}_h), \quad \forall \bar{u}_h \in U_h, \quad (4.5.8)$$

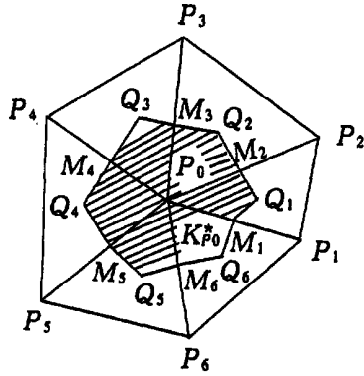


Fig. 4.5.1

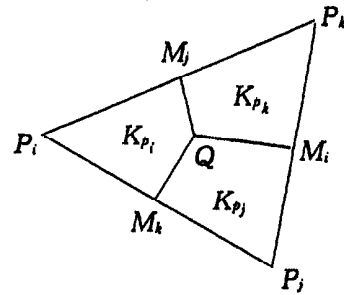


Fig. 4.5.2

4.5.2 Error estimate

Next we analyze the linear element scheme. Write $A(w; u_h, \Pi_h^* \bar{u}_h)$ into the following form:

$$A(w; u_h, \Pi_h^* \bar{u}_h) = \sum_{K \in T_h} I_K(w; u_h, \Pi_h^* \bar{u}_h), \quad (4.5.9)$$

where $K = \triangle P_i P_j P_k$ (cf. Fig. 4.5.2), and

$$\begin{aligned} & I_K(w; u_h, \Pi_h^* \bar{u}_h) \\ &= (\bar{u}_h(P_j) - \bar{u}_h(P_i)) \int_{\overline{M_k Q}} a(w) \frac{\partial u_h}{\partial n} ds \\ & \quad + (\bar{u}_h(P_k) - \bar{u}_h(P_j)) \int_{\overline{M_i Q}} a(w) \frac{\partial u_h}{\partial n} ds \\ & \quad + (\bar{u}_h(P_i) - \bar{u}_h(P_k)) \int_{\overline{M_j Q}} a(w) \frac{\partial u_h}{\partial n} ds \\ &= \sum_{l=i,j,k} |\overline{P_{l+1} P_{l+2}}| \int_{\overline{M_l Q}} a(w) \frac{\partial u_h}{\partial n_l} \frac{\partial \bar{u}_h}{\partial \tau_l} ds, \end{aligned} \quad (4.5.10)$$

where n_l denotes the unit outer normal vector of $K_{P_{l+2}}$ along $\overline{M_l Q}$, and $\tau_l = \overline{P_{l+1} P_{l+2}} / |\overline{P_{l+1} P_{l+2}}|$. ($l = i, j, k$; $i+1 = j, j+1 = k, k+1 = i$.) Now we are ready to deduce the boundedness and the positive definiteness of $A(w; u_h, \Pi_h^* \bar{u}_h)$.

Theorem 4.5.1 *For the bilinear form $A(w; \cdot, \Pi_h^* \cdot)$, there exists a constant C independent of U_h such that*

$$|A(w; u_h, \Pi_h^* \bar{u}_h)| \leq C \|u_h\|_1 \|\bar{u}_h\|_1, \quad \forall u_h, \bar{u}_h \in U_h, \quad (4.5.11)$$

$$|A(w; u, \Pi_h^* \bar{u}_h)| \leq C \|u\|_{1,\infty} \|\bar{u}_h\|_1, \quad \forall u \in W^{1,\infty}(\Omega), \bar{u}_h \in U_h. \quad (4.5.12)$$

Moreover, if T_h^* is a circumcenter dual grid, then we have a constant $\beta > 0$ independent of U_h satisfying

$$A(w; u_h, \Pi_h^* u_h) \geq \beta \|u_h\|_1^2, \quad \forall u_h \in U_h. \quad (4.5.13)$$

This also implies the existence of a solution to the linear element difference scheme (4.5.6).

Proof (4.5.11) follows from (4.5.9), (4.5.10) and the equivalent norms defined in §3.2 :

$$\begin{aligned}
& |A(w; u_h, \Pi_h^* \bar{u}_h)| \\
& \leq C \sum_{K \in T_h} h^2 \left(\left| \frac{\partial u_h(Q)}{\partial x} \right| + \left| \frac{\partial u_h(Q)}{\partial y} \right| \right) \left(\left| \frac{\partial \bar{u}_h(Q)}{\partial x} \right| + \left| \frac{\partial \bar{u}_h(Q)}{\partial y} \right| \right) \\
& \leq C \|u_h\|_1 \|\bar{u}_h\|_1.
\end{aligned}$$

(4.5.12) can be similarly proved. In the case of the circumcenter dual grid, we have $n_l = \tau_l$. Hence by use of (4.5.10) we have

$$\begin{aligned}
& I_K(w; u_h, \Pi_h^* u_h) \\
& = \sum_{l=i,j,k} |\bar{P}_{l+1} \bar{P}_{l+2}| \int_{\bar{M}_l Q} a(w) \left(\frac{\partial u_h}{\partial \tau_l} \right)^2 ds \\
& \geq \beta' S_K \left(\left(\frac{\partial u_h(Q)}{\partial x} \right)^2 + \left(\frac{\partial u_h(Q)}{\partial y} \right)^2 \right). \quad (\beta' > 0.)
\end{aligned}$$

This gives (4.5.13).

To show the solvability of (4.5.6) we define $T : U_h \rightarrow U_h$ by

$$A(w_h; Tw_h, \Pi_h^* \bar{u}_h) = (f, \Pi_h^* \bar{u}_h), \quad \forall \bar{u}_h \in U_h. \quad (4.5.14)$$

By virtue of (4.5.11) and (4.5.13), we have the existence and the uniqueness of the solution Tw_h as well as the estimate

$$\|Tw_h\|_1 \leq \|f\|_0 / \beta.$$

Thus the mapping T maps the ball $\{w_h \in U_h : \|w_h\|_1 \leq \|f\|_0 / \beta\}$ to itself. Also note that T is obviously a continuous mapping. Therefore the Brouwer fixed point theorem guarantees the existence of the solution $u_h \in U_h$ and the estimate

$$\|u_h\|_1 \leq \|f\|_0 / \beta.$$

This completes the proof. $\square$

Theorem 4.5.2 *Let u be the solution of the weak form (4.5.3) of the problem (4.5.1), and $u_h \in U_h$ the solution of the linear element generalized difference scheme (4.5.6). If $u \in W^{2,\infty}(\Omega)$, then there holds the following error estimate:*

$$\|u - u_h\|_1 \leq Ch.$$

The proof to this theorem is omitted. For the details, see [B-58].

Bibliography and Comments

This chapter provides some main results on the mixed and the non-conforming generalized difference methods for the boundary value problem of the fourth order elliptic equation. As regards the mixed method, [A-51] gives a mixed generalized difference method for biharmonic equations, based on a Ciarlet-Raviart mixed variational principle. Another mixed generalized difference method is presented in [A-31] based on a Hermann-Miyoshi mixed variational principle. But there are some errors in the proofs of the error estimations in both the two papers. A refined error analysis is provided in §4.1 (Theorem 4.1.2). In order to construct nonconforming generalized difference methods for biharmonic equations, a corresponding variational principle is discussed and is used to give a nonconforming generalized difference scheme with Zienkiewicz elements [A-9]. Another nonconforming generalized difference method is proposed in [A-8], based on Adini elements. The error analysis and numerical experiments are carried out in these two papers. Theoretical analysis indicates that these nonconforming methods enjoy the same error estimate as the corresponding finite element methods. Both the mixed and the nonconforming generalized difference methods require less computational time than the finite element methods, and have better accuracy than the usual finite difference methods. We also note that it is easy for them to deal with complex boundaries and various of boundary conditions.

With regard to nonlinear elliptic equations, [B-58] constructed a generalized difference scheme for a second order nonlinear Dirichlet

problem, and presented corresponding error estimates.

Problem The cubic generalized difference method has a stronger non-conformity since the piecewise cubic element space U_h generally is not contained in H^2 when the dimension $n \geq 2$. This brings new difficulties for the construction of the difference scheme and for the error estimation. But if we adopt the Hsieh-Clough-Tocher triangular elements (cf. [A-27,2] and [p.340, B-17]), then we indeed have $U_h \in H^2$. Try to use such a U_h (and a certain corresponding V_h) to construct generalized difference schemes and to deduce the error estimates.

Chapter 5

PARABOLIC EQUATIONS

We present in this chapter, for second order parabolic differential equations, semi- and fully-discrete generalized difference methods and one of their varieties – a mass concentration method. The construction as well as the theoretical analysis of these schemes are discussed. A nonlinear parabolic equation is considered in the last section.

5.1 Semi-discrete Generalized Difference Schemes

5.1.1 Problem and schemes

Consider the parabolic differential equation:

$$\begin{cases} u_t + Au = f(x, t), & x \in \Omega, 0 < t \leq T, & (5.1.1a) \\ u = 0, & x \in \partial\Omega, 0 < t \leq T, & (5.1.1b) \\ u = u_0(x), & x \in \Omega, t = 0, & (5.1.1c) \end{cases}$$

where Ω is a bounded region in R^n , with a Lipschitz continuous boundary; $u_t = \frac{\partial u}{\partial t}$; and A a second order elliptic differential operator:

$$Au \equiv - \sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + \sum_j b_j \frac{\partial u}{\partial x_j} + cu,$$

where $a_{ij}(=a_{ji})$, b_j and c are sufficiently smooth functions of x . We assume there is a constant $\alpha_0 > 0$ satisfying

$$\begin{aligned} a(u, u) &= \int_{\Omega} \left(\sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_j b_j \frac{\partial u}{\partial x_j} u + cu^2 \right) dx \\ &\geq \alpha_0 \|u\|_1^2, \quad \forall u \in H_0^1(\Omega). \end{aligned} \quad (5.1.2)$$

The variational problem related to (5.1.1) is: Find $u = u(\cdot, t) \in H_0^1(\Omega)$ ($0 \leq t \leq T$) such that

$$\begin{cases} (u_t, v) + a(u, v) = (f, v), & \forall v \in H_0^1(\Omega), t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (5.1.3a)$$

$$(5.1.3b)$$

where $(\cdot, \cdot)$ denotes the inner product of $L^2(\Omega)$, and

$$a(u, v) = \int_{\Omega} \left(\sum_{i,j} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_j b_j \frac{\partial u}{\partial x_j} v + cuv \right) dx. \quad (5.1.4)$$

The solution to (5.1.3) is called the generalized solution of (5.1.1).

As in Chapters 2 and 3, we place a quasi-uniform grid and a corresponding dual grid on $\bar{\Omega}$, and construct a trial function space $U_h \in H_0^1(\Omega)$ and a test function space $V_h \in L^2(\Omega)$. Then the semi-discrete generalized difference scheme for (5.1.1) is: Find $u_h = u_h(\cdot, t) \in U_h$ ($0 \leq t \leq T$) such that

$$\begin{cases} (u_{h,t}, v_h) + a(u_h, v_h) = (f, v_h), & \forall v_h \in V_h, t > 0, \\ u_h(x, 0) = u_{0h}(x), & x \in \Omega, \end{cases} \quad (5.1.5a)$$

$$(5.1.5b)$$

where $a(u_h, v_h)$ is a bilinear form obtained by applying piecewise Green's formula to (Au, v) ; or by interpreting the right-hand side of (5.1.4) in the sense of generalized functions, namely, the integrals are computed in terms of a δ -function method on the boundaries of neighbouring dual elements. u_{0h} is a certain approximation of u_0 on U_h . A commonly used method is to choose u_{0h} as an interpolation projection of u_0 in U_h . Another way is to replace (5.1.5b) by

$$(u_h(\cdot, 0), v_h) = (u_0, v_h), \quad \forall v_h \in V_h.$$

Let $\{\phi_j(x) : j = 1, 2, \dots, n\}$ and $\{\psi_j(x) : j = 1, 2, \dots, n\}$ be the bases of U_h and V_h respectively. Then (5.1.5) can be expressed as: Find a solution of the form

$$u_h = \sum_{j=1}^n \mu_j(t) \phi_j(x),$$

such that its coefficients $\mu_1(t), \mu_2(t), \dots, \mu_n(t)$ satisfy

$$\begin{cases} \sum_{j=1}^n \left[\frac{d\mu_j(t)}{dt} (\phi_j, \psi_i) + \mu_j(t) a(\phi_j, \phi_i) \right] = (f, \psi_i), & t > 0, \\ i = 1, 2, \dots, n, & (5.1.5a)' \\ \mu_j(0) = \alpha_j, & j = 1, 2, \dots, n, & (5.1.5b)' \end{cases}$$

where α_j 's are the coefficients in $u_{0h} = \sum_{j=1}^n \alpha_j \phi_j$.

Let us introduce the following matrix and vector notations:

$$M = [m_{ij}] = [(\phi_j, \psi_i)], \quad K = [k_{ij}] = [a(\phi_j, \psi_i)],$$

$$\mathbf{u} = [\mu_1(t), \dots, \mu_n(t)]^T, \quad F = [(f, \psi_1), \dots, (f, \psi_n)]^T,$$

$$\alpha = [\alpha_1, \dots, \alpha_n]^T.$$

Then we can rewrite (5.1.5)' as

$$\begin{cases} M \frac{d\mathbf{u}}{dt} + K\mathbf{u} = F, & (5.1.5a)'' \\ \mathbf{u}(0) = \alpha. & (5.1.5b)'' \end{cases}$$

As in the finite element method, we call M a mass matrix, and K a stiff matrix. M is clearly nonsingular. The ordinary differential equation theory tells us that this semi-discrete generalized difference scheme has a unique solution for any $f \in L^2(\Omega)$.

We are mainly concerned in this chapter with two-dimensional problems. So we always assume that Ω is a planar polygonal region, and that A is a second order elliptic differential operator:

$$Au \equiv - \left[\frac{\partial}{\partial x} \left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) \right] + qu.$$

We also assume $a_{ij}(x, y)$'s ($i, j = 1, 2$) and $q(x, y)$ are sufficiently smooth and positive definite: There exists a constant $r > 0$ such that

$$\sum_{i,j=1}^2 a_{ij}(x, y) \xi_i \xi_j \geq r \sum_{i=1}^2 \xi_i^2, \quad q(x, y) \geq 0,$$

$$\forall (\xi_i, \xi_j) \in \mathbb{R}^2, (x, y) \in \bar{\Omega}.$$

In this and the next two sections, we always assume that U_h is a piecewise linear function space corresponding to a grid T_h of $\bar{\Omega}$, and that V_h is a piecewise constant function space with respect to the barycenter dual grid T_h^* . (cf. §3.2 for details.)

5.1.2 Some lemmas

First let us restate some results of the last two chapters.

Lemma 5.1.1 *Set*

$$\begin{aligned} \|u_h\|_{0,h} &= \|\Pi_h^* u_h\|_0 = \left\{ \sum_{K_{P_0}^* \in T_h^*} u_h^2(P_0) S_{P_0}^* \right\}^{1/2} \\ &= \left\{ \frac{1}{3} \sum_{K_Q \in T_h} [u_h^2(P_i) + u_h^2(P_j) + u_h^2(P_k)] S_Q \right\}^{1/2}, \end{aligned} \quad (5.1.6)$$

$$|u_h|_{1,h} = \left\{ \sum_{K_Q \in T_h} \left[\left(\frac{\partial u_h(Q)}{\partial x} \right)^2 + \left(\frac{\partial u_h(Q)}{\partial y} \right)^2 \right] S_Q \right\}^{1/2}, \quad (5.1.7)$$

$$\|u_h\|_{1,h} = \{\|u_h\|_{0,h}^2 + |u_h|_{1,h}^2\}^{1/2}. \quad (5.1.8)$$

Then on U_h the following pairs of norms are equivalent respectively: $|\cdot|_{1,h}$ and $\|\cdot\|_1$; $\|\cdot\|_{0,h}$ and $\|\cdot\|_0$; and $\|\cdot\|_{1,h}$ and $\|\cdot\|_1$.

Lemma 5.1.2 *The bilinear form $a(u_h, \Pi_h^* \bar{u}_h)$ can be expressed as*

$$a(u_h, \Pi_h^* \bar{u}_h) = a_h(u_h, \Pi_h^* \bar{u}_h) + b_h(u_h, \Pi_h^* \bar{u}_h), \quad (5.1.9)$$

where the leading term

$$\begin{aligned}
 & a_h(u_h, \Pi_h^* \bar{u}_h) \\
 = & \sum_{K_Q \in T_h} \left\{ \left[a_{11}(Q) \frac{\partial u_h(Q)}{\partial x} + a_{12}(Q) \frac{\partial u_h(Q)}{\partial y} \right] \frac{\partial \bar{u}_h(Q)}{\partial x} \right. \\
 & + \left. \left[a_{21}(Q) \frac{\partial u_h(Q)}{\partial x} + a_{22}(Q) \frac{\partial u_h(Q)}{\partial y} \right] \frac{\partial \bar{u}_h(Q)}{\partial y} \right\} S_Q \\
 & + \sum_{K_{P_0}^* \in T_h^*} q(P_0) u_h(P_0) \bar{u}_h(P_0) S_{P_0}^*
 \end{aligned} \tag{5.1.10}$$

is symmetric and positive definite (c_1 and c_2 are positive constants):

$$a_h(u_h, \Pi_h^* \bar{u}_h) = a_h(\bar{u}_h, \Pi_h^* u_h), \quad \forall \bar{u}_h, u_h \in U_h, \tag{5.1.11}$$

$$c_1 \|u_h\|_1^2 \leq a_h(u_h, \Pi_h^* u_h) \leq c_2 \|u_h\|_1^2, \quad \forall u_h \in U_h, \tag{5.1.12}$$

and the remainder $b_h(u_h, \Pi_h^* \bar{u}_h)$ satisfies

$$|b_h(u_h, \Pi_h^* \bar{u}_h)| \leq Ch \|u_h\|_1 \|\bar{u}_h\|_1, \quad \forall \bar{u}_h, u_h \in U_h. \tag{5.1.13}$$

If we define $\|u_h\|_1 = [a_h(u_h, \Pi_h^* u_h)]^{1/2}$, then $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent on U_h (cf. (5.1.12)). We also have (see (5.1.9), (5.1.11) and (5.1.13))

$$\begin{aligned}
 & |a(u_h, \Pi_h^* \bar{u}_h) - a(\bar{u}_h, \Pi_h^* u_h)| \\
 & \leq Ch \|u_h\|_1 \|\bar{u}_h\|_1, \quad \forall \bar{u}_h, u_h \in U_h.
 \end{aligned} \tag{5.1.14}$$

Lemma 5.1.3 *There exist positive constants h_0, α and M such that when $0 < h \leq h_0$*

$$a(u_h, \Pi_h^* u_h) \geq \alpha \|u_h\|_1^2, \quad \forall u_h \in U_h, \tag{5.1.15}$$

$$|a(u_h, \Pi_h^* \bar{u}_h)| \leq M \|u_h\|_1 \|\bar{u}_h\|_1, \quad \forall \bar{u}_h, u_h \in U_h. \tag{5.1.16}$$

Lemma 5.1.4 *Let $u \in H_0^1(\Omega)$ be the solution to the variational problem*

$$a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega)$$

and $u_h \in U_h$ to the generalized difference scheme

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h,$$

then we have

$$\|u - u_h\|_1 \leq Ch|u|_2, \quad (5.1.17)$$

$$\|u - u_h\|_0 \leq Ch^2 \|u\|_{3,p}, \quad (p > 1) \quad (5.1.18)$$

Lemma 5.1.5 *There hold the following statements:*

$$(i) (u_h, \Pi_h^* \bar{u}_h) = (\bar{u}_h, \Pi_h^* u_h), \quad \forall \bar{u}_h, u_h \in U_h. \quad (5.1.19)$$

(ii) Set $|||u_h|||_0 = (u_h, \Pi_h^* u_h)^{1/2}$. Then $|||\cdot|||_0$ is equivalent to $\|\cdot\|_0$ on U_h , that is, there exist positive constants c_3 and c_4 such that

$$c_3 \|u_h\|_0 \leq |||u_h|||_0 \leq c_4 \|u_h\|_0, \quad \forall u_h \in U_h. \quad (5.1.20)$$

The above results can be found in §3.2 and §4.1.

Let us introduce an elliptic projection operator

$$P_h : H^2(\Omega) \cap H_0^1(\Omega) \rightarrow U_h,$$

defined by the following generalized difference scheme:

$$a(P_h u, v_h) = a(u, v_h), \quad \forall v_h \in V_h. \quad (5.1.21)$$

By Lemma 5.1.3 we see that $P_h u$ is uniquely defined by (5.1.21) for any $u \in H^2(\Omega) \cap H_0^1(\Omega)$. We call $P_h u$ the elliptic projection of u (with respect to the generalized difference scheme). By Lemma 5.1.4 we have the following estimate.

Lemma 5.1.6 *Let $P_h u$ be the elliptic projection of u defined by (5.1.21), then*

$$\|u - P_h u\|_1 \leq Ch|u|_2, \quad (5.1.22)$$

$$\|u - P_h u\|_0 \leq Ch^2 \|u\|_{3,p}, \quad (p > 1) \quad (5.1.23)$$

5.1.3 L^2 -error estimate

Theorem 5.1.1 *Let u and u_h be the solutions to the problem (5.1.3) and the semi-discrete generalized difference scheme (5.1.5), respectively. Then we have*

$$\begin{aligned} & \|u - u_0\|_0 \\ & \leq C \left\{ \|u_0 - u_{0h}\|_0 + h^2 \left[\|u_0\|_{3,p} + \int_0^t \|u_\tau\|_{3,p} d\tau \right] \right\}, \quad (p > 1) \end{aligned} \quad (5.1.24)$$

Proof Write

$$\rho = u - P_h u, \quad e = P_h u - u_h, \quad (5.1.25)$$

where P_h is the elliptic projection operator. Then we have

$$u - u_h = \rho + e. \quad (5.1.26)$$

It follows from (5.1.23) that

$$\begin{aligned} \|\rho\|_0 & \leq Ch^2 \|u\|_{3,p} = Ch^2 \|u_0 + \int_0^t u_\tau d\tau\|_{3,p} \\ & \leq Ch^2 \left[\|u_0\|_{3,p} + \int_0^t \|u_\tau\|_{3,p} d\tau \right]. \end{aligned} \quad (5.1.27)$$

We turn to estimate e . Since u and u_h satisfy (5.1.3) and (5.1.5) respectively, we have

$$(u_t - u_{h,t}, v_h) + a(u - u_h, v_h) = 0, \quad \forall v_h \in V_h. \quad (5.1.28)$$

This together with (5.1.21) gives

$$(e_t, v_h) + a(e, v_h) = -(\rho_t, v_h), \quad \forall v_h \in V_h. \quad (5.1.29)$$

Choosing $v_h = \Pi_h^* e$ and using (5.1.19) and (5.1.15) yield

$$\frac{1}{2} \frac{d}{dt} \|e\|_0^2 \leq \|\rho_t\|_0 \|\Pi_h^* e\|_0.$$

So it follows from Lemmas 5.1.1 and 5.1.5 that

$$\frac{d}{dt} \|e\|_0 \leq C \|\rho_t\|_0.$$

Integrate it with respect to t and note the equivalence of the norms $||| \cdot |||_0$ and $\| \cdot \|_0$, then we have

$$\|e\|_0 \leq C \left[\|e(0)\|_0 + \int_0^t \|\rho_\tau\|_0 d\tau \right]. \quad (5.1.30)$$

By virtue of Lemma 5.1.6 we have

$$\begin{aligned} \|e(0)\|_0 &\leq \|P_h u_0 - u_0\|_0 + \|u_0 - u_{0h}\|_0 \\ &\leq Ch^2 \|u_0\|_{3,p} + \|u_0 - u_{0h}\|_0, \end{aligned} \quad (5.1.31)$$

$$\|\rho_\tau\|_0 = \|u_\tau - P_h u_\tau\|_0 \leq Ch^2 \|u_\tau\|_{3,p}. \quad (5.1.32)$$

A combination of (5.1.27) and (5.1.30)-(5.1.32) leads to (5.1.24). This completes the proof. $\square$

5.1.4 H^1 -error estimate

Theorem 5.1.2 *Let u and u_h be the solutions to the problem (5.1.3) and the semi-discrete generalized difference scheme (5.1.5), respectively. Then we have*

$$\begin{aligned} &\|u - u_h\|_1 \\ &\leq C \left\{ \|u_0 - u_{0h}\|_1 + h \left[\|u_0\|_2 + \int_0^t \|u_\tau\|_2 d\tau + \left(\int_0^t \|u_\tau\|_2^2 d\tau \right)^{1/2} \right] \right\}. \end{aligned} \quad (5.1.33)$$

Proof Take $v_h = \Pi_h^* e_t$ in (5.1.29) to get

$$\begin{aligned} &|||e_t|||_0^2 + a(e, \Pi_h^* e_t) = -(\rho_t, \Pi_h^* e_t), \\ &|||e_t|||_0^2 + \frac{1}{2} \frac{d}{dt} a(e, \Pi_h^* e) \\ &= -(\rho_t, \Pi_h^* e_t) + \frac{1}{2} [a(e_t, \Pi_h^* e) - a(e, \Pi_h^* e_t)]. \end{aligned} \quad (5.1.34)$$

It follows from (5.1.14) and the inverse property of the finite element space that

$$\begin{aligned} |a(e_t, \Pi_h^* e) - a(e, \Pi_h^* e_t)| &\leq Ch \|e_t\|_1 \|e\|_1 \\ &\leq C \|e_t\|_0 \|e\|_1 \leq \|e_t\|_0^2 + C' \|e\|_1^2. \end{aligned}$$

Thus

$$\frac{d}{dt} a(e, \Pi_h^* e) \leq C (\|\rho_t\|_0^2 + \|e\|_1^2).$$

Integrate on t and use Lemma 5.1.3 to obtain

$$\begin{aligned} \alpha \|e\|_1^2 &\leq a(e, \Pi_h^* e) \\ &\leq a(e(0), \Pi_h^* e(0)) + C \int_0^t (\|\rho_\tau\|_0^2 + \|e\|_1^2) d\tau \\ &\leq M \|e(0)\|_1^2 + C \int_0^t (\|\rho_\tau\|_0^2 + \|e\|_1^2) d\tau. \end{aligned}$$

Note

$$\begin{aligned} \|e(0)\|_1 &\leq \|P_h u_0 - u_0\|_1 + \|u_0 - u_{0h}\|_1 \\ &\leq Ch \|u_0\|_2 + \|u_0 - u_{0h}\|_1. \end{aligned}$$

Hence

$$\|e\|_1^2 \leq C \left\{ \|u_0 - u_{0h}\|_1^2 + h^2 \|u_0\|_2^2 + \int_0^t (\|\rho_\tau\|_0^2 + \|e\|_1^2) d\tau \right\}.$$

By the well-known Gronwall inequality we have

$$\|e\|_1^2 \leq C \left\{ \|u_0 - u_{0h}\|_1^2 + h^2 \|u_0\|_2^2 + \int_0^t \|\rho_\tau\|_0^2 d\tau \right\}. \quad (5.1.35)$$

By virtue of Lemma 5.1.6 we have

$$\begin{aligned} \|\rho\|_1 &= \|u - P_h u\|_1 \leq Ch \|u\|_2 \\ &\leq Ch (\|u_0\|_2 + \int_0^t \|u_\tau\|_2 d\tau), \end{aligned} \quad (5.1.36)$$

$$\|\rho_\tau\|_0 = \|u_\tau - P_h u_\tau\|_0 \leq Ch \|u_\tau\|_2. \quad (5.1.37)$$

Inserting (5.1.37) into (5.1.35) yields

$$\begin{aligned} \|\varepsilon\|_1 \leq & C \left\{ \|u_0 - u_{0h}\|_1 + h \|u_0\|_2 \right. \\ & \left. + h \left(\int_0^t \|u_\tau\|_2^2 d\tau \right)^{1/2} \right\}. \end{aligned} \quad (5.1.38)$$

A combination of (5.1.36) and (5.1.38) leads to (5.1.33). This completes the proof. $\square$

5.2 Fully-discrete Generalized Difference Schemes

5.2.1 Fully-discrete schemes

In the last section the semi-discrete schemes are obtained by discretizing the space variable. In order to finally get numerical solutions we need to further discretize the time variable to obtain fully-discrete schemes. To this end, there are two methods most in use: the implicit Euler's scheme (backward differencing) and the Crank-Nicolson scheme (central differencing).

Let τ denote the time step size, and $t_n = n\tau$ ($n = 0, 1, \dots$), $u_h^n = u_h(t_n)$. At time $t = t_n$, if we use the backward difference quotient

$$\bar{\partial}_t u_h^n = (u_h^n - u_h^{n-1})/\tau$$

to approximate the differential quotient $u_{h,t}$ in the semi-discrete scheme, then we obtain a fully-discrete scheme: Find $u_h^n \in U_h$ ($n = 1, 2, \dots$) such that

$$\begin{cases} (\bar{\partial}_t u_h^n, v_h) + a(u_h^n, v_h) = (f(t_n), v_h), \quad \forall v_h \in V_h, & (5.2.1a) \\ n = 1, 2, \dots, \\ u_h^0 = u_{0h}. & (5.2.1b) \end{cases}$$

Or we can equivalently write it as

$$\begin{cases} (u_h^n, v_h) + \tau a(u_h^n, v_h) = (u_h^{n-1} + \tau f(t_n), v_h), \quad \forall v_h \in V_h, \\ n = 1, 2, \dots, \\ u_h^0 = u_{0h}. \end{cases}$$

This scheme is referred to as a backward Euler generalized difference scheme.

By (5.1.15)

$$\begin{aligned} & a(u_h^n, \Pi_h^* u_h^n) + \frac{1}{\tau} (u_h^n, \Pi_h^* u_h^n) \\ & \geq \alpha \|u_h^n\|_1^2, \quad \forall u_h^n \in U_h. \end{aligned}$$

This guarantees the existence and uniqueness of the solution u_h^n to (5.2.1a) for a given u_h^{n-1} .

If we discretize the semi-discrete scheme at time $t_{n-1/2} = (n - \frac{1}{2})\tau$ in a symmetric fashion, then we have another fully-discrete scheme as follows: Find $u_h^n \in U_h$ ($n = 1, 2, \dots$) such that

$$\begin{cases} (\bar{\partial}_t u_h^n, v_h) + a\left(\frac{u_h^n + u_h^{n-1}}{2}, v_h\right) = \\ \quad \left(\frac{f(t_n) + f(t_{n-1})}{2}, v_h\right), \quad \forall v_h \in V_h, \quad (5.2.2a) \\ n = 1, 2, \dots, \\ u_h^0 = u_{0h}. \end{cases} \quad (5.2.2b)$$

This scheme is the so-called Crank-Nicolson generalized difference scheme. The existence and uniqueness of its solution can be readily proved similarly as above.

5.2.2 Error estimates for backward Euler generalized difference schemes

Theorem 5.2.1 Let u and $\{u_h^n\}$ be the solutions to the parabolic equation (5.1.3) and the backward Euler generalized difference scheme

(5.2.1), respectively. Then

$$\begin{aligned} & \|u(t_n) - u_h^n\|_0 \\ & \leq C \left\{ \|u_0 - u_{0h}\|_0 + h^2 \left[\|u_0\|_{3,p} + \int_0^{t_n} \|u_t\|_{3,p} dt \right] \right. \\ & \quad \left. + \tau \int_0^{t_n} \|u_{tt}\|_0 dt \right\}, \quad n = 1, 2, \dots \quad (p > 1) \end{aligned} \quad (5.2.3)$$

Proof Set

$$\rho^n = u(t_n) - P_h u(t_n), \quad e^n = P_h u(t_n) - u_h^n,$$

then

$$u(t_n) - u_h^n = \rho^n + e^n. \quad (5.2.4)$$

It follows from (5.1.23) that

$$\|\rho^n\|_0 \leq Ch^2 \|u(t_n)\|_{3,p} \leq Ch^2 \left[\|u_0\|_{3,p} + \int_0^{t_n} \|u_t\|_{3,p} dt \right]. \quad (5.2.5)$$

Set $t = t_n$ in (5.1.3), and subtract it and (5.2.1), then we have

$$(u_t(t_n) - \bar{\partial}_t u_h^n, v_h) + a(\rho^n + e^n, v_h) = 0, \quad \forall v_h \in V_h. \quad (5.2.6)$$

By virtue of (5.1.21) we have

$$\begin{aligned} & (\bar{\partial}_t e^n, v_h) + a(e^n, v_h) \\ & = (\bar{\partial}_t P_h u(t_n) - u_t(t_n), v_h), \quad \forall v_h \in V_h. \end{aligned} \quad (5.2.7)$$

Write $r^n = \bar{\partial}_t P_h u(t_n) - u_t(t_n)$, set $v_h = \Pi_h^* e^n$, and use (5.1.15), then we have

$$(\bar{\partial}_t e^n, \Pi_h^* e^n) \leq (r^n, \Pi_h^* e^n).$$

So

$$\begin{aligned} |||e^n|||_0^2 & \leq (e^{n-1}, \Pi_h^* e^n) + \tau (r^n, \Pi_h^* e^n), \\ |||e^n|||_0^2 & \leq (|||e^{n-1}|||_0 + \tau |||r^n|||_0) |||e^n|||_0. \end{aligned}$$

Eliminating $|||e^n|||_0$ and using the above recursion relation, we have

$$|||e^n|||_0 \leq |||e^0|||_0 + \tau \sum_{j=1}^n |||r^j|||_0.$$

Making use of the equivalence of the norms we get

$$\|e^n\|_0 \leq C(\|e^0\|_0 + \tau \sum_{j=1}^n \|r^j\|_0). \quad (5.2.8)$$

Write $r^j = r_1^j + r_2^j$, where

$$\begin{aligned} r_1^j &= \bar{\partial}_t P_h u(t_j) - \bar{\partial}_t u(t_j) = \frac{1}{\tau} \int_{t_{j-1}}^{t_j} (P_h - I) u_t dt, \\ r_2^j &= \bar{\partial}_t u(t_j) - u_t(t_j) = -\frac{1}{\tau} \int_{t_{j-1}}^{t_j} (t - t_{j-1}) u_{tt} dt. \end{aligned}$$

Then by (5.1.23)

$$\begin{aligned} \sum_{j=1}^n \|r_1^j\|_0 &\leq \frac{1}{\tau} \sum_{j=1}^n \int_{t_{j-1}}^{t_j} C h^2 \|u_t\|_{3,p} dt \\ &= C \tau^{-1} h^2 \int_0^{t_n} \|u_t\|_{3,p} dt. \end{aligned} \quad (5.2.9)$$

Similarly

$$\sum_{j=1}^n \|r_2^j\|_0 \leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|u_{tt}\|_0 dt = \int_0^{t_n} \|u_{tt}\|_0 dt. \quad (5.2.10)$$

Again by (5.1.23)

$$\begin{aligned} \|e^0\|_0 &\leq \|P_h u_0 - u_0\|_0 + \|u_0 - u_{0h}\|_0 \\ &\leq C h^2 \|u_0\|_{3,p} + \|u_0 - u_{0h}\|_0. \end{aligned} \quad (5.2.11)$$

Substituting (5.2.9)-(5.2.11) into (5.2.8) yields

$$\begin{aligned} \|e^n\|_0 &\leq C\{\|u_0 - u_{0h}\|_0 + h^2[\|u_0\|_{3,p} \\ &\quad + \int_0^{t_n} \|u_t\|_{3,p} dt] + \tau \int_0^{t_n} \|u_{tt}\|_0 dt\}. \end{aligned} \quad (5.2.12)$$

Finally (5.2.3) follows from (5.2.5) and (5.2.12). This completes the proof. $\square$

Next we deal with the H^1 -estimate.

Theorem 5.2.2 *Let u and $\{u_h^n\}$ be the solutions to the parabolic equation (5.1.9) and the backward Euler generalized difference scheme respectively. Then*

$$\begin{aligned} & \|u(t_n) - u_h^n\|_1 \\ & \leq C \left\{ \|u_0 - u_{0h}\|_1 + h \left[\|u_0\|_2 + \int_0^{t_n} \|u_t\|_2 dt \right. \right. \\ & \quad \left. \left. + \left(\int_0^{t_n} \|u_t\|_2^2 dt \right)^{1/2} \right] + \tau \left(\int_0^{t_n} \|u_{tt}\|_0^2 dt \right)^{1/2} \right\}, \\ & \quad n = 1, 2, \dots \end{aligned} \quad (5.2.13)$$

Proof As in the proof to Theorem 5.2.1, we can again obtain (5.2.7). To proceed, we set $v_h = \Pi_h^* \bar{\partial}_t e^n$ to get

$$|||\bar{\partial}_t e^n|||_0^2 + a(e^n, \Pi_h^* \bar{\partial}_t e^n) = (r^n, \Pi_h^* \bar{\partial}_t e^n). \quad (5.2.14)$$

By the equivalence of the norms we have a constant $C_0 > 0$ satisfying

$$|||\bar{\partial}_t e^n|||_0^2 \geq C_0 \|\bar{\partial}_t e^n\|_0^2. \quad (5.2.15)$$

It follows from Lemmas 5.1.2 and 5.1.3 that

$$\begin{aligned} & a(e^n, \Pi_h^* \bar{\partial}_t e^n) \\ & = \frac{1}{2\tau} [a(e^n + e^{n-1}, \Pi_h^*(e^n - e^{n-1})) \\ & \quad + a(e^n - e^{n-1}, \Pi_h^*(e^n - e^{n-1}))] \\ & \geq \frac{1}{2\tau} [a_h(e^n + e^{n-1}, \Pi_h^*(e^n - e^{n-1})) \\ & \quad + b_h(e^n + e^{n-1}, \Pi_h^*(e^n - e^{n-1}))] \\ & \geq \frac{1}{2\tau} [|||e^n|||_1^2 - |||e^{n-1}|||_1^2] \\ & \quad - C \|e^n + e^{n-1}\|_1 \|\bar{\partial}_t e^n\|_0 \\ & \geq \frac{1}{2\tau} [(1 - C'\tau) |||e^n|||_1^2 - (1 + C'\tau) |||e^{n-1}|||_1^2] \\ & \quad - \frac{C_0}{2} \|\bar{\partial}_t e^n\|_0^2. \end{aligned} \quad (5.2.16)$$

Also note

$$|(r^n, \Pi_h^* \bar{\partial}_t e^n)| \leq C \|r^n\|_0^2 + \frac{C_0}{2} \|\bar{\partial}_t e^n\|_0^2. \quad (5.2.17)$$

Combining (5.2.14)-(5.2.17) gives

$$\|e^n\|_1^2 \leq \frac{1 + C'\tau}{1 - C'\tau} \|e^{n-1}\|_1^2 + C\tau \|r^n\|_0^2. \quad (5.2.18)$$

This recursion relation leads to

$$\|e^n\|_1^2 \leq C \left(\|e^0\|_1^2 + \tau \sum_{j=1}^n \|r^j\|_0^2 \right). \quad (5.2.19)$$

Note

$$\begin{aligned} r^j &= r_1^j + r_2^j, \\ r_1^j &= \bar{\partial}_t P_h u(t_j) - \bar{\partial}_t u(t_j) = \frac{1}{\tau} \int_{t_{j-1}}^{t_j} (P_h u_t - u_t) dt, \\ r_2^j &= \bar{\partial}_t u(t_j) - u_t(t_j) = -\frac{1}{\tau} \int_{t_{j-1}}^{t_j} (t - t_{j-1}) u_{tt} dt. \end{aligned}$$

So we have

$$\begin{aligned} \sum_{j=1}^n \|r_1^j\|_0^2 &\leq C\tau^{-2} \sum_{j=1}^n \left(\int_{t_{j-1}}^{t_j} h \|u_t\|_2 dt \right)^2 \\ &\leq C\tau^{-1} h^2 \int_0^{t_n} \|u_t\|_2^2 dt, \end{aligned} \quad (5.2.20)$$

$$\begin{aligned} \sum_{j=1}^n \|r_2^j\|_0^2 &\leq \sum_{j=1}^n \left(\int_{t_{j-1}}^{t_j} \|u_{tt}\|_0 dt \right)^2 \\ &\leq \tau \int_0^{t_n} \|u_{tt}\|_0^2 dt. \end{aligned} \quad (5.2.21)$$

Also notice

$$\begin{aligned} \|e^0\|_1^2 &\leq \|P_h u_0 - u_0\|_1^2 + \|u_0 - u_{0h}\|_1^2 \\ &\leq Ch^2 \|u_0\|_2^2 + \|u_0 - u_{0h}\|_1^2. \end{aligned} \quad (5.2.22)$$

A combination of (5.2.19)-(5.2.22) yields

$$\begin{aligned} \|e^n\|_1 \leq & C \left[\|u_0 - u_{0h}\|_1 + h\|u_0\|_2 + h \left(\int_0^{t_n} \|u_t\|_2^2 dt \right)^{1/2} \right. \\ & \left. + \tau \left(\int_0^{t_n} \|u_{tt}\|_0^2 dt \right)^{1/2} \right]. \end{aligned} \quad (5.2.23)$$

On the other hand,

$$\begin{aligned} \|\rho^n\|_1 &= \|u(t_n) - P_h u(t_n)\|_1 \leq Ch \|u(t_n)\|_2 \\ &\leq Ch \left[\|u_0\|_2 + \int_0^{t_n} \|u_t\|_2 dt \right]. \end{aligned} \quad (5.2.24)$$

Finally, (5.2.13) results from (5.2.23) and (5.2.24). This completes the proof. $\square$

5.2.3 Error estimates for Crank-Nicolson generalized difference schemes

Theorem 5.2.3 *Let u and $\{u_h^n\}$ be the solutions to the parabolic problem (5.1.3) and the Crank-Nicolson generalized difference scheme (5.2.2), respectively, then*

$$\begin{aligned} & \|u(t_n) - u_h^n\|_0 \\ \leq & C \left\{ \|u_0 - u_{0h}\|_0 + h^2 \left[\|u_0\|_{3,p} + \int_0^{t_n} \|u_t\|_{3,p} dt \right] \right. \\ & \left. + \tau^2 \int_0^{t_n} \|u_{ttt}\|_0 dt \right\}, \quad n = 1, 2, \dots \quad (p > 1) \end{aligned} \quad (5.2.25)$$

Proof As before we set

$$u(t_n) - u_h^n = \rho^n + e^n, \quad (5.2.26)$$

where

$$\rho^n = u(t_n) - P_h u(t_n), \quad e^n = P_h u(t_n) - u_h^n.$$

For ρ^n we have

$$\begin{aligned} \|\rho^n\|_0 &\leq Ch^2 \|u(t_n)\|_{3,p} \\ &\leq Ch^2 \left[\|u_0\|_{3,p} + \int_0^{t_n} \|u_t\|_{3,p} dt \right]. \end{aligned} \quad (5.2.27)$$

On the other hand, by (5.1.3), (5.1.21) and (5.2.2), e^n satisfies

$$(\bar{\partial}_t e^n, v_h) + a\left(\frac{e^n + e^{n-1}}{2}, v_h\right) = (r^n, v_h), \quad \forall v_h \in V_h, \quad (5.2.28)$$

where

$$r^n = \bar{\partial}_t P_h u(t_n) - \frac{u_t(t_n) + u_t(t_{n-1})}{2}.$$

Take $v_h = \Pi_h^* \frac{e^n + e^{n-1}}{2}$ in (5.2.28) to get

$$\left(\bar{\partial}_t e^n, \Pi_h^* \frac{e^n + e^{n-1}}{2}\right) \leq \left(r^n, \Pi_h^* \frac{e^n + e^{n-1}}{2}\right).$$

By virtue of Lemma 5.1.5 we have

$$\frac{1}{2\tau} (|||e^n|||_0^2 - |||e^{n-1}|||_0^2) \leq \frac{1}{2} |||r^n|||_0 (|||e^n|||_0 + |||e^{n-1}|||_0).$$

Thus,

$$|||e^n|||_0 \leq |||e^{n-1}|||_0 + C\tau |||r^n|||_0.$$

By this recursion relation we have

$$|||e^n|||_0 \leq |||e^0|||_0 + C\tau \sum_{j=1}^n |||r^j|||_0,$$

and hence

$$||e^n||_0 \leq C \left(||e^0||_0 + \tau \sum_{j=1}^n ||r^j||_0 \right). \quad (5.2.29)$$

Write

$$\begin{aligned} r^j &= r_1^j + r_2^j, \\ r_1^j &= \bar{\partial}_t P_h u(t_j) - \bar{\partial}_t u(t_j) = \frac{1}{\tau} \int_{t_{j-1}}^{t_j} (P_h u_t - u_t) dt, \\ r_2^j &= \bar{\partial}_t u(t_j) - \frac{u_t(t_j) + u_t(t_{j-1})}{2}. \end{aligned}$$

By Lemma 5.1.6

$$\sum_{j=1}^n ||r_1^j||_0 \leq C\tau^{-1}h^2 \int_0^{t_n} \|u_t\|_{3,p} dt. \quad (5.2.30)$$

By the Taylor expansion we have

$$\sum_{j=1}^n \|r_2^j\|_0 \leq C\tau \int_0^{t_n} \|u_{ttt}\|_0 dt. \quad (5.2.31)$$

Also note

$$\begin{aligned} \|e^0\|_0 &\leq \|P_h u_0 - u_0\|_0 + \|u_0 - u_{0h}\|_0 \\ &\leq Ch^2 \|u_0\|_{3,p} + \|u_0 - u_{0h}\|_0. \end{aligned} \quad (5.2.32)$$

This together with (5.2.29)-(5.2.31) gives

$$\begin{aligned} \|e^n\|_0 &\leq C \left\{ \|u_0 - u_{0h}\|_0 + h^2 [\|u_0\|_{3,p} \right. \\ &\quad \left. + \int_0^{t_n} \|u_t\|_{3,p} dt] + \tau^2 \int_0^{t_n} \|u_{ttt}\|_0 dt \right\}. \end{aligned} \quad (5.2.33)$$

Now, (5.2.25) follows from (5.2.27) and (5.2.33), This completes the proof. $\square$

Theorem 5.2.4 *Let u and $\{u_h^n\}$ be the solutions to the parabolic problem (5.1.9) and the Crank-Nicolson generalized difference scheme (5.2.2), respectively, then*

$$\begin{aligned} &\|u(t_n) - u_h^n\|_1 \\ &\leq C \left\{ \|u_0 - u_{0h}\|_1 + h \left[\|u_0\|_2 + \int_0^{t_n} \|u_t\|_2 dt \right. \right. \\ &\quad \left. \left. + \left(\int_0^{t_n} \|u_t\|_2^2 dt \right)^{1/2} \right] + \tau^2 \left(\int_0^{t_n} \|u_{ttt}\|_0^2 dt \right)^{1/2} \right\}, \\ &n = 1, 2, \dots \end{aligned} \quad (5.2.34)$$

Proof As in the proof of the last theorem, we have

$$(\bar{\partial}_t e^n, v_h) + a \left(\frac{e^n + e^{n-1}}{2}, v_h \right) = (r^n, v_h), \quad \forall v_h \in V_h. \quad (5.2.35)$$

Choosing $v_h = \Pi_h^* \bar{\partial}_t e^n$ leads to

$$\|\bar{\partial}_t e^n\|_0^2 + a \left(\frac{e^n + e^{n-1}}{2}, \Pi_h^* \bar{\partial}_t e^n \right) = (r^n, \Pi_h^* \bar{\partial}_t e^n). \quad (5.2.36)$$

As in the proof to Theorem 5.2.2, we use Lemma 5.2.2 and the inverse property of the finite element space U_h , and note the equivalence of the norms to obtain

$$\begin{aligned}
& a\left(\frac{e^n + e^{n-1}}{2}, \Pi_h^* \bar{\partial}_t e^n\right) \\
&= \frac{1}{2\tau} [a_h(e^n + e^{n-1}, \Pi_h^*(e^n - e^{n-1})) \\
&\quad + b_h(e^n + e^{n-1}, \Pi_h^*(e^n - e^{n-1}))] \\
&\geq \frac{1}{2\tau} [|||e^n|||_1^2 - |||e^{n-1}|||_1^2] \\
&\quad - C\|e^n + e^{n-1}\|_1 \|\bar{\partial}_t e^n\|_0 \\
&\geq \frac{1}{2\tau} [(1 - C'\tau) |||e^n|||_1^2 - (1 + C'\tau) |||e^{n-1}|||_1^2] \\
&\quad - \frac{1}{2} \|\bar{\partial}_t e^n\|_0^2,
\end{aligned} \tag{5.2.37}$$

$$|(r^n, \Pi_h^* \bar{\partial}_t e^n)| \leq C\|r^n\|_0^2 + \frac{1}{2} |||\bar{\partial}_t e^n|||_0^2. \tag{5.2.38}$$

It follows from (5.2.36)-(5.2.38) that

$$|||e^n|||_1^2 \leq \frac{1 + C'\tau}{1 - C'\tau} |||e^{n-1}|||_1^2 + C\tau\|r^n\|_0^2.$$

This implies

$$|||e^n|||_1^2 \leq C\left(|||e^0|||_1^2 + \tau \sum_{j=1}^n \|r^j\|_0^2\right). \tag{5.2.39}$$

Now, similar to (5.2.22) we have

$$\|e^0\|_1^2 \leq Ch^2\|u_0\|_2^2 + \|u_0 - u_{0h}\|_1^2. \tag{5.2.40}$$

As for $r^j = r_1^j + r_2^j$, we imitate (5.2.30) and (5.2.31) to get

$$\begin{aligned}
\sum_{j=1}^n \|r_1^j\|_0^2 &\leq C\tau^{-2}h^2 \sum_{j=1}^n \left(\int_{t_{j-1}}^{t_j} \|u_t\|_2 dt\right)^2 \\
&\leq C\tau^{-1}h^2 \int_0^{t_n} \|u_t\|_2^2 dt,
\end{aligned} \tag{5.2.41}$$

$$\begin{aligned}
\sum_{j=1}^n \|r_2^j\|_0^2 &\leq C\tau^2 \sum_{j=1}^n \left(\int_{t_{j-1}}^{t_j} \|u_{ttt}\|_0 dt \right)^2 \\
&\leq C\tau^3 \int_0^{t_n} \|u_{ttt}\|_0^2 dt.
\end{aligned} \tag{5.2.42}$$

Combining (5.2.39)-(5.2.42) yields

$$\begin{aligned}
\|e^n\|_1 &\leq C \left\{ \|u_0 - u_{0h}\|_1 + h \left[\|u_0\|_2 + \left(\int_0^{t_n} \|u_t\|_2^2 dt \right)^{1/2} \right] \right. \\
&\quad \left. + \tau^2 \left(\int_0^{t_n} \|u_{ttt}\|_0^2 dt \right)^{1/2} \right\}.
\end{aligned} \tag{5.2.43}$$

Similar to (5.2.24) we have

$$\|\rho^n\|_1 \leq Ch \left[\|u_0\|_2 + \int_0^{t_n} \|u_t\|_2 dt \right]. \tag{5.2.44}$$

Finally, (5.2.34) results from (5.2.26), (5.2.43) and (5.2.44). This completes the proof. $\square$

5.3 Mass Concentration Methods

This section is devoted to a variety of the generalized difference method, a mass concentration method, for parabolic equations. This method simplifies the computation and enjoys a satisfactory convergence.

5.3.1 Construction of schemes

Let us recall the semi-discrete generalized difference scheme (5.1.5):

$$\begin{cases} (u_{h,t}, v_h) + a(u_h, v_h) = (f, v_h), & \forall v_h \in V_h, t > 0, \\ u_h(x, 0) = u_{0h}(x), & x \in \Omega. \end{cases} \tag{5.3.1a}$$

$$\tag{5.3.1b}$$

Its equivalent matrix form is:

$$\begin{cases} M \frac{\partial \mathbf{u}}{\partial t} + K \mathbf{u} = F, \\ \mathbf{u}(0) = \alpha, \end{cases} \tag{5.3.1a}'$$

$$\tag{5.3.1b}'$$

where M is a mass matrix, and K a stiff matrix.

The idea of the so-called mass concentration method is to concentrate all the entries on each row of the mass matrix $M = [m_{ij}]$ to the diagonal position, such that the inverse of M is extremely easy to get and hence greatly simplifies the computation. To elaborate, the scheme of semi-discrete mass concentration method is:

$$\begin{cases} \bar{M} \frac{\partial \mathbf{u}}{\partial t} + K \mathbf{u} = F, \\ \mathbf{u}(0) = \alpha. \end{cases} \quad (5.3.2a)$$

$$(5.3.2b)$$

where $\bar{M} = [\bar{m}_{ij}]$ is a diagonal matrix

$$\bar{m}_{ij} = \begin{cases} 0, & \text{when } j \neq i, \\ \sum_{k=1}^n m_{ik}, & \text{when } j = i. \end{cases} \quad (5.3.3)$$

Now, we deduce an equivalent form of the above scheme, which will be used later on for error estimates.

Define a semi-discrete problem: Find $u_h = \sum_{j=1}^n \mu_j(t) \phi_j \in U_h$ such that

$$\begin{cases} (\Pi_h^* u_{h,t}, v_h) + a(u_h, v_h) = (f, v_h), & v_h \in V_h, t > 0, \\ u_h(x, 0) = u_{0,h}(x), & x \in \Omega. \end{cases} \quad (5.3.4a)$$

$$(5.3.4b)$$

Lemma 5.3.1 *Problems (5.3.2) and (5.3.4) are equivalent.*

Proof Write (5.3.4) into a matrix form:

$$\begin{cases} \tilde{M} \frac{d\mathbf{u}}{dt} + K \mathbf{u} = F, \\ \mathbf{u}(0) = \alpha. \end{cases}$$

Apparently K , f and α here are identical to those in (5.3.2). It merely remains to show $\tilde{M} = \bar{M}$. The ij -entry of $\tilde{M}$ is

$$\tilde{m}_{ij} = (\Pi_h^* \phi_j, \psi_i) = (\psi_j, \psi_i) = \begin{cases} 0, & \text{when } j \neq i, \\ S_{P_i}^*, & \text{when } j = i. \end{cases}$$

Here ψ_i is the characteristic function of the dual element $K_{P_i}^*$, and $S_{P_i}^*$ is the area of $K_{P_i}^*$. By (5.3.3)

$$\tilde{m}_{ij} = \begin{cases} 0, & \text{when } j \neq i, \\ \sum_{k=1}^n (\phi_k, \psi_i) = (1, \psi_i) = S_{P_i}^*, & \text{when } j = i. \end{cases}$$

Thus $\tilde{M} = \overline{M}$. This completes the proof. $\square$

The fully-discrete mass concentration scheme is: Find $u_h^n \in U_h$ ($n = 1, 2, \dots$) such that

$$\begin{cases} (\Pi_h^* \bar{\partial}_t u_h^n, v_h) + a(\theta u_h^n + (1 - \theta)u_h^{n-1}, v_h) \\ \quad = (\theta f(t_n) + (1 - \theta)f(t_{n-1}), v_h), & (5.3.5a) \\ v_h \in V_h, n = 1, 2, \dots, \\ u_h^0 = u_{0h}. & (5.3.5b) \end{cases}$$

Its matrix form is

$$\begin{cases} (\overline{M} + \theta \tau K) \mathbf{u}^n \\ \quad = [\overline{M} - (1 - \theta) \tau K] \mathbf{u}^{n-1} + \tau [\theta f^n + (1 - \theta) f^{n-1}], \\ \quad n = 1, 2, \dots, \\ \mathbf{u}^0 = \alpha. \end{cases}$$

This leads to a backward Euler fully-discrete scheme of the mass concentration method when $\theta = 1$, and a Crank-Nicolson scheme when $\theta = \frac{1}{2}$.

5.3.2 Error estimates for semi-discrete schemes

Theorem 5.3.1 *Let u and u_h be the solutions to the problem (5.1.9) and the semi-discrete, mass concentration, generalized difference scheme (5.3.4), respectively. Then we have*

$$\begin{aligned} & \|u - u_h\|_1 \\ & \leq C \left\{ \|u_0 - u_{0h}\|_1 + h \left[\|u_0\|_2 + \int_0^t \|u_\tau\|_2 d\tau + \left(\int_0^t \|u_\tau\|_2^2 d\tau \right)^{1/2} \right] \right\}. \end{aligned} \quad (5.3.6)$$

Proof As in §5.2, we write

$$u - u_h = \rho + e, \quad \rho = u - P_h u, \quad e = P_h u - u_h, \quad (5.3.7)$$

where P_h is the elliptic projection operator. By (5.1.22)

$$\|\rho\|_1 \leq Ch\|u\|_2 \leq Ch\left(\|u_0\|_2 + \int_0^t \|u_\tau\|_2 d\tau\right). \quad (5.3.8)$$

Since u and u_h satisfy (5.1.3) and (5.3.4) respectively, we have

$$(u_t - \Pi_h^* u_{h,t}, v_h) + a(u - u_h, v_h) = 0, \quad v_h \in V_h. \quad (5.3.9)$$

This together with (5.1.21) gives

$$(\Pi_h^* e_t, v_h) + a(e, v_h) = -(r, v_h),$$

where

$$r = u_t - \Pi_h^* P_h u_t.$$

Set $v_h = \Pi_h^* e_t$ and use Lemma 5.1.2 to obtain

$$\|e_t\|_{0,h}^2 + a_h(e, \Pi_h^* e_t) = -b_h(e, \Pi_h^* e_t) - (r, \Pi_h^* e_t).$$

We have the following estimates for the above terms.

$$\|e_t\|_{0,h}^2 \geq C_0 \|e_t\|_0^2,$$

$$a_h(e, \Pi_h^* e_t) = \frac{1}{2} \frac{d}{dt} \|e\|_1^2,$$

$$\begin{aligned} |b_h(e, \Pi_h^* e_t)| &\leq Ch \|e\|_1 \|e_t\|_1 \leq C \|e\|_1 \|e_t\|_0 \\ &\leq C \|e\|_1^2 + \frac{C_0}{2} \|e_t\|_0^2, \end{aligned}$$

$$|(r, \Pi_h^* e_t)| \leq C \|r\|_0^2 + \frac{C_0}{2} \|e_t\|_0^2.$$

Therefore,

$$\frac{d}{dt} \|e\|_1^2 \leq C(\|e\|_1^2 + \|r\|_0^2).$$

Integrate on t and note

$$\|e(0)\|_1 \leq Ch\|u_0\|_2 + \|u_0 - u_{0h}\|_1,$$

then we have

$$\begin{aligned} \|e\|_1^2 \leq & C[\|u_0 - u_{0h}\|_1^2 + h^2\|u_0\|_2^2 \\ & + \int_0^t (\|e\|_1^2 + \|r\|_0^2) d\tau]. \end{aligned}$$

Make use of Gronwall's inequality to get

$$\|e\|_1^2 \leq C(\|u_0 - u_{0h}\|_1^2 + h^2\|u_0\|_2^2 + \int_0^t \|r\|_0^2 d\tau). \quad (5.3.10)$$

Write

$$\begin{aligned} r &= r_1 + r_2, \\ r_1 &= u_t - P_h u_t, \quad r_2 = P_h u_t - \Pi_h^* P_h u_t. \end{aligned}$$

Then it is easy to see that

$$\|r_1\|_0 \leq Ch\|u_t\|_2,$$

$$\|r_2\|_0 \leq Ch\|P_h u_t\|_1 \leq Ch\|u_t\|_2.$$

Inserting the above two estimates into (5.3.10) yields

$$\|e\|_1 \leq C[\|u_0 - u_{0h}\|_1 + h\|u_0\|_2 + h(\int_0^t \|u_\tau\|_2^2 d\tau)^{1/2}]. \quad (5.3.11)$$

Finally, (5.3.6) follows from (5.3.7), (5.3.8) and (5.3.11). This completes the proof. $\square$

5.3.3 Error estimates for fully-discrete schemes

Theorem 5.3.2 *Let u and $\{u_h^n\}$ be the solutions to the parabolic equation (5.1.3) and the backward Euler, mass concentration, generalized difference scheme (5.3.5), respectively. Then*

$$\begin{aligned} & \|u(t_n) - u_h^n\|_1 \\ & \leq C\left\{\|u_0 - u_{0h}\|_1 + h\left[\|u_0\|_2 + \int_0^{t_n} \|u_t\|_2 dt\right.\right. \\ & \quad \left.\left.+ \left(\int_0^{t_n} \|u_t\|_2^2 dt\right)^{1/2}\right] + \tau\left(\int_0^{t_n} \|u_{tt}\|_0^2 dt\right)^{1/2}\right\}, \\ & \quad n = 0, 1, 2, \dots \end{aligned} \quad (5.3.12)$$

Proof Write

$$u(t_n) - u_h^n = \rho^n + e^n, \quad (5.3.13)$$

where

$$\rho^n = u(t_n) - P_h u(t_n), \quad e^n = P_h u(t_n) - u_h^n.$$

It is obvious that

$$\|\rho^n\|_1 \leq Ch \|u(t_n)\|_2 \leq Ch \left[\|u_0\|_2 + \int_0^{t_n} \|u_t\|_2 dt \right]. \quad (5.3.14)$$

It is easy to check that e^n satisfies

$$(\Pi_h^* \bar{\partial}_t e^n, v_h) + a(e^n, v_h) = (r^n, v_h), \quad v_h \in V_h, \quad (5.3.15)$$

where

$$r^n = \Pi_h^* \bar{\partial}_t P_h u(t_n) - u_t(t_n). \quad (5.3.16)$$

Setting $v_h = \Pi_h^* \bar{\partial}_t e^n$ yields

$$\|\bar{\partial}_t e^n\|_{0,h}^2 + a(e^n, \Pi_h^* \bar{\partial}_t e^n) = (r^n, \Pi_h^* \bar{\partial}_t e^n).$$

We have the following estimates for the above terms.

$$\begin{aligned} \|\bar{\partial}_t e^n\|_{0,h}^2 &\geq C_0 \|\bar{\partial}_t e^n\|_0^2, \\ a(e^n, \Pi_h^* \bar{\partial}_t e^n) &\geq \frac{1}{2\tau} [a_h(e^n + e^{n-1}, \Pi_h^*(e^n - e^{n-1})) \\ &\quad + b_h(e^n + e^{n-1}, \Pi_h^*(e^n - e^{n-1}))] \\ &\geq \frac{1}{2\tau} [(1 - C\tau) \|e^n\|_1^2 - (1 + C\tau) \|e^{n-1}\|_1^2] \\ &\quad - \frac{C_0}{2} \|\bar{\partial}_t e^n\|_0^2, \\ |(r^n, \Pi_h^* \bar{\partial}_t e^n)| &\leq C \|r^n\|_0^2 + \frac{C_0}{2} \|\bar{\partial}_t e^n\|_0^2. \end{aligned}$$

Consequently

$$\|e^n\|_1^2 \leq \frac{1 + C\tau}{1 - C\tau} \|e^{n-1}\|_1^2 + C\tau \|r^n\|_0^2.$$

This implies

$$\|e^n\|_1^2 \leq C \left(\|e^0\|_1^2 + \tau \sum_{j=1}^n \|r^j\|_0^2 \right). \quad (5.3.17)$$

Set

$$r^j = r_0^j + r_1^j + r_2^j,$$

where

$$r_0^j = \Pi_h^* \bar{\partial}_t P_h u(t_j) - \bar{\partial}_t P_h u(t_j) = (\Pi_h^* - I) \tau^{-1} \int_{t_{j-1}}^{t_j} P_h u_t dt,$$

$$r_1^j = \bar{\partial}_t P_h u(t_j) - \bar{\partial}_t u(t_j), \quad r_2^j = \bar{\partial}_t u(t_j) - u_t(t_j).$$

Correspondingly we have the following estimates:

$$\begin{aligned} \sum_{j=1}^n \|r_0^j\|_0^2 &\leq Ch^2 \tau^{-2} \sum_{j=1}^n \left(\int_{t_{j-1}}^{t_j} \|P_h u_t\|_2^2 dt \right)^2 \\ &\leq Ch^2 \tau^{-1} \int_0^{t_n} \|u_t\|_2^2 dt, \end{aligned}$$

$$\sum_{j=1}^n \|r_1^j\|_0^2 \leq Ch^2 \tau^{-1} \int_0^{t_n} \|u_t\|_2^2 dt,$$

$$\sum_{j=1}^n \|r_2^j\|_0^2 \leq \tau \int_0^{t_n} \|u_{tt}\|_0^2 dt.$$

Substituting these estimates into (5.3.17) gives

$$\begin{aligned} \|e^n\|_1 &\leq C \left\{ \|u_0 - u_{0h}\|_1 + h \left[\|u_0\|_2 + \left(\int_0^{t_n} \|u_t\|_2^2 dt \right)^{1/2} \right] \right. \\ &\quad \left. + \tau \left(\int_0^{t_n} \|u_{tt}\|_0^2 dt \right)^{1/2} \right\}. \end{aligned} \quad (5.3.18)$$

A combination of (5.3.13), (5.3.14) and (5.3.18) leads to (5.3.12). This completes the proof. $\square$

5.4 High Order Element Difference Schemes

This section is concerned with high order element difference schemes for parabolic equations. First we discuss a cubic element difference scheme for parabolic equations in one dimension, and present its error estimate. Then we consider a quadratic element difference scheme for parabolic equations in two dimensions, for which a numerical example is also provided.

5.4.1 Cubic element difference schemes for one-dimensional parabolic equations

Consider the mixed problem of the one-dimensional parabolic equation:

$$\begin{cases} \frac{\partial u}{\partial t} + Lu = f(x, t), & x \in (a, b), 0 < t \leq T, \\ u(a, t) = 0, \frac{\partial u(b, t)}{\partial x} = 0, & 0 < t \leq T, \\ u(x, 0) = u_0(x), & x \in (a, b), \end{cases} \quad \begin{matrix} (5.4.1a) \\ (5.4.1b) \\ (5.4.1c) \end{matrix}$$

where

$$Lu \equiv -\frac{\partial}{\partial x} \left(p \frac{\partial u}{\partial x} \right) + r \frac{\partial u}{\partial x} + qu,$$

$p \in C^1[a, b]$, $p \geq p_{\min} > 0$, $q, r \in C[a, b]$, and $f \in L^2(a, b)$.

Let us place a quasi-uniform grid T_h and a corresponding barycenter dual grid T_h^* on $[a, b]$. Take the trial function space U_h as the Hermite cubic element space related to T_h , and the test function space V_h as the piecewise linear function space with respect to T_h^* . For details, see §2.4.

The cubic element semi-discrete difference scheme reads: Find $u_h = u_h(\cdot, t) \in U_h$ ($0 < t \leq T$) such that

$$\begin{cases} \left(\frac{\partial u_h}{\partial t}, v_h \right) + (Lu_h, v_h) = (f, v_h), \\ \forall v_h \in V_h, 0 < t \leq T, \\ u_h(x, 0) = u_{0h}(x), \quad x \in (a, b), \end{cases} \quad \begin{matrix} (5.4.2a) \\ (5.4.2b) \end{matrix}$$

where $u_{0h} \in U_h$ is some approximation of u_0 .

The cubic element fully-discrete difference scheme is: Find $u_h^n \in U_h$ ($n = 1, 2, \dots$) such that

$$\begin{cases} (\bar{\partial}_t u_h^n, v_h) + (Lu_h^{n,\theta}, v_h) = (f^{n,\theta}, v_h), \\ \forall v_h \in V_h, n = 1, 2, \dots, \\ u_h^0 = u_{0h}, \end{cases} \quad (5.4.3a)$$

$$(5.4.3b)$$

where (τ is the time step size, and $t_n = n\tau$)

$$\bar{\partial}_t u_h^n = \frac{u_h^n - u_h^{n-1}}{\tau}, \quad u_h^{n,\theta} = \theta u_h^n + (1 - \theta) u_h^{n-1},$$

$$f^{n,\theta} = \theta f^n + (1 - \theta) f^{n-1}, \quad f^n = f(t_n).$$

(5.4.3) leads to a backward Euler fully-discrete scheme when $\theta = 1$, and a Crank-Nicolson fully-discrete scheme when $\theta = \frac{1}{2}$.

The following lemmas will be used later on for the error estimates.

Lemma 5.4.1 *The elliptic projection $P_h u \in U_h$ of $u \in H^2(\Omega) \cap H_0^1(\Omega)$ is uniquely defined by*

$$(LP_h u, v_h) = (Lu, v_h), \quad v_h \in V_h, \quad (5.4.4)$$

and satisfies

$$\|P_h u - u\|_m \leq Ch^{4-m} |u|_4, \quad m = 0, 1. \quad (5.4.5)$$

Proof The conclusion is a consequence of Theorems 2.4.1, 2.4.3 and 2.5.2. $\square$

Lemma 5.4.2 *Let $\Pi_h^* u_h$ denote the interpolation projection of u_h onto V_h . Then, $(Lu_h, \Pi_h^* \bar{u}_h)$ can be expressed as*

$$(Lu_h, \Pi_h^* \bar{u}_h) = a_1(u_h, \Pi_h^* \bar{u}_h) + a_2(u_h, \Pi_h^* \bar{u}_h), \quad (5.4.6)$$

where the leading term satisfies (c_1 and c_2 are positive constants)

$$a_1(u_h, \Pi_h^* \bar{u}_h) = a_1(\bar{u}_h, \Pi_h^* u_h), \quad \forall \bar{u}_h, u_h \in U_h, \quad (5.4.7)$$

$$c_1 \|u_h\|_1^2 \leq a_1(u_h, \Pi_h^* u_h) \leq c_2 \|u_h\|_1^2, \quad \forall u_h \in U_h, \quad (5.4.8)$$

and the remainder term satisfies

$$|a_2(u_h, \Pi_h^* \bar{u}_h)| \leq ch \|u_h\|_1 \|\bar{u}\|_1, \quad \forall \bar{u}_h, u_h \in U_h. \quad (5.4.9)$$

Set

$$|||u_h|||_1 = [a_1(u_h, \Pi_h^* u_h)]^{1/2}, \quad u_h \in U_h, \quad (5.4.10)$$

then $||| \cdot |||_1$ is equivalent to the H^1 -norm $\| \cdot \|_1$.

Proof Identify $a_1(u_h, \Pi_h^* \bar{u}_h)$ with $b_h(u_h, \Pi_h^* \bar{u}_h)$ in §2.4. Imitating the proofs to (2.4.13), (2.4.17) and (2.4.18), we can show (5.4.7), (5.4.8) and (5.4.9) respectively. This completes the proof. $\square$

A straightforward calculation verifies the following lemma.

Lemma 5.4.3 *There exists a constant $\beta > 0$ independent of the subspace U_h such that*

$$(u_h, \Pi_h^* u_h) \geq \beta \|u_h\|_0^2, \quad \forall u_h \in U_h. \quad (5.4.11)$$

Theorem 5.4.1 *Let u and u_h be the solutions to the problem (5.4.1) and the semi-discrete cubic element generalized difference scheme (5.4.2), respectively. Then we have*

$$\begin{aligned} \|u - u_h\|_1 \leq & C \left\{ \|u_0 - u_{0h}\|_1 + h^3 [\|u_0\|_4 \right. \\ & \left. + \int_0^t \|u_\tau\|_4 d\tau + h \left(\int_0^t \|u_\tau\|_4^2 d\tau \right)^{1/2}] \right\}. \end{aligned} \quad (5.4.12)$$

Proof Set

$$u - u_h = \rho + e, \quad (5.4.13)$$

where

$$\rho = u - P_h u, \quad e = P_h u - u_h.$$

By Lemma 5.4.1 we have

$$\|\rho\|_1 \leq Ch^3 \|u\|_4 \leq Ch^3 \left(\|u_0\|_4 + \int_0^t \|u_\tau\|_4 d\tau \right). \quad (5.4.14)$$

Since u and u_h satisfy (5.4.1) and (5.4.2) respectively, we have

$$\left(\frac{\partial u}{\partial t} - \frac{\partial u_h}{\partial t}, v_h\right) + (Lu - Lu_h, v_h) = 0, \quad \forall v_h \in V_h. \quad (5.4.15)$$

Thus it follows from (5.4.4) that

$$(e_t, v_h) + (Le, v_h) = -(\rho_t, v_h), \quad \forall v_h \in V_h. \quad (5.4.16)$$

Choosing $v_h = \Pi_h^* e_t$ leads to

$$(e_t, \Pi_h^* e_t) + a_1(e, \Pi_h^* e_t) = -(\rho_t, \Pi_h^* e_t) - a_2(e, \Pi_h^* e_t).$$

Now, it results from (5.4.11), (5.4.7), (5.4.10) and the inverse property of the finite element space that

$$\begin{aligned} & \beta \|e_t\|_0^2 + \frac{1}{2} \frac{d}{dt} \|e\|_1^2 \\ & \leq \|\rho_t\|_0 \|e_t\|_0 + C \|e\|_1 \|e_t\|_0 \\ & \leq C(\|\rho_t\|_0^2 + \|e\|_1^2) + \beta \|e_t\|_0^2. \end{aligned}$$

Simplify it and integrate it, then we have

$$\|e\|_1^2 \leq \|e(0)\|_1^2 + C \int_0^t (\|\rho_t\|_0^2 + \|e\|_1^2) d\tau. \quad (5.4.17)$$

Notice the equivalence of the norms and the inequality

$$\begin{aligned} \|e(0)\|_1 & \leq \|P_h u_0 - u_0\|_1 + \|u_0 - u_{0h}\|_1 \\ & \leq Ch^3 \|u_0\|_4 + \|u_0 - u_{0h}\|_1. \end{aligned}$$

Hence it follows from the Gronwall's inequality that

$$\|e\|_1^2 \leq C \left\{ \|u_0 - u_{0h}\|_1^2 + h^6 \|u_0\|_4^2 + \int_0^t \|\rho_\tau\|_0^2 d\tau \right\}. \quad (5.4.18)$$

By Lemma 5.4.1 we have

$$\|\rho_\tau\|_0 \leq Ch^4 \|u_\tau\|_4. \quad (5.4.19)$$

(5.4.18) and (5.4.19) imply

$$\begin{aligned} \|e\|_1 \leq & C \left\{ \|u_0 - u_{0h}\|_1 + h^3 \|u_0\|_4 \right. \\ & \left. + h^4 \left(\int_0^t \|u_\tau\|_4^2 d\tau \right)^{1/2} \right\}. \end{aligned} \quad (5.4.20)$$

A combination of (5.4.13), (5.4.14) and (5.4.20) implies (5.4.12). This completes the proof. $\square$

Theorem 5.4.2 *Let u and $\{u_h^n\}$ be the solutions to the parabolic equation (5.4.1) and the Crank-Nicolson cubic element generalized difference scheme (5.4.3), respectively. Then*

$$\begin{aligned} & \|u(t_n) - u_h^n\|_1 \\ \leq & C \left\{ \|u_0 - u_{0h}\|_1 + h^3 \left[\|u_0\|_4 + \int_0^{t_n} \|u_t\|_4 dt \right. \right. \\ & \left. \left. + h \left(\int_0^{t_n} \|u_t\|_4^2 dt \right)^{1/2} \right] + \tau^2 \left(\int_0^{t_n} \|u_{ttt}\|_0^2 dt \right)^{1/2} \right\}. \end{aligned} \quad (5.4.21)$$

Proof By (5.4.1) and (5.4.3)

$$\left(\frac{\partial u}{\partial t}, v_h \right) + (Lu, v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad (5.4.22)$$

$$(\bar{\partial}_t u_h^n, v_h) + \left(L \frac{u_h^n + u_h^{n-1}}{2}, v_h \right) = \left(\frac{f^n + f^{n-1}}{2}, v_h \right), \quad \forall v_h \in V_h. \quad (5.4.23)$$

Set $t = t_n$ and $t = t_{n-1}$ respectively in (5.4.22), combine them with (5.4.23) and use (5.4.4), then we have

$$(\bar{\partial}_t e^n, v_h) + \left(L \frac{e^n + e^{n-1}}{2}, v_h \right) = (r^n, v_h), \quad \forall v_h \in V_h, \quad (5.4.24)$$

where

$$e^n = P_h u(t_n) - u_h^n, \quad r^n = \bar{\partial}_t P_h u(t_n) - \frac{\partial}{\partial t} \frac{u(t_n) + u(t_{n-1})}{2}.$$

Choosing $v_h = \Pi_h^* \bar{\partial}_t e^n$ in (5.4.24) yields

$$\begin{aligned} & (\bar{\partial}_t e^n, \Pi_h^* \bar{\partial}_t e^n) + \frac{1}{2\tau} [a_1(e^n + e^{n-1}, \Pi_h^*(e^n - e^{n-1})) \\ & + a_2(e^n + e^{n-1}, \Pi_h^*(e^n - e^{n-1}))] = (r^n, \Pi_h^* \bar{\partial}_t e^n). \end{aligned}$$

It follows from (5.4.11), (5.4.7), (5.4.10) and (5.4.9) that

$$\begin{aligned} & \beta \|\bar{\partial}_t e^n\|_0^2 + \frac{1}{2\tau} (\|e^n\|_1^2 - \|e^{n-1}\|_1^2) - C \|e^n + e^{n-1}\|_1 \|\bar{\partial}_t e^n\|_0 \\ & \leq C \|r^n\|_0 \|\bar{\partial}_t e^n\|_0. \end{aligned}$$

So there is a constant $C' > 0$ such that

$$\begin{aligned} & \beta \|\bar{\partial}_t e^n\|_0^2 + \frac{1}{2\tau} [(1 - C'\tau) (\|e^n\|_1^2 \\ & - (1 + C'\tau) \|e^{n-1}\|_1^2) - \frac{\beta}{2} \|\bar{\partial}_t e^n\|_0^2] \\ & \leq C \|r^n\|_0^2 + \frac{\beta}{2} \|\bar{\partial}_t e^n\|_0^2. \end{aligned}$$

Thus

$$\|e^n\|_1^2 \leq \frac{1 + C'\tau}{1 - C'\tau} \|e^{n-1}\|_1^2 + C\tau \|r^n\|_0^2.$$

This recursion relation implies the existence of a constant $C > 0$ such that

$$\|e^n\|_1^2 \leq C \left(\|e^0\|_1^2 + \tau \sum_{j=1}^n \|r^j\|_0^2 \right). \quad (5.4.25)$$

By virtue of (5.4.5) we have

$$\begin{aligned} \|e^0\|_1 & \leq C \|e^0\|_1 = C \|P_h u^0 - u_{0h}\|_1 \\ & \leq C (\|u_0 - P_h u_0\|_1 + \|u_0 - u_{0h}\|_1) \\ & \leq Ch^3 \|u_0\|_4 + C \|u_0 - u_{0h}\|_1. \end{aligned} \quad (5.4.26)$$

As before, we write

$$\begin{aligned} r^j & = r_1^j + r_2^j, \\ r_1^j & = \bar{\partial}_t P_h u(t_j) - \bar{\partial}_t u(t_j) = \frac{1}{\tau} \int_{t_{j-1}}^{t_j} (P_h u_t - u_t) dt, \end{aligned} \quad (5.4.27)$$

$$r_2^j = \bar{\partial}_t u(t_j) - \frac{u_t(t_j) + u_t(t_{j-1})}{2}.$$

So by (5.4.5) we have

$$\begin{aligned} \sum_{j=1}^n \|r_1^j\|_0^2 &\leq C\tau^{-2} \sum_{j=1}^n \left(\int_{t_{j-1}}^{t_j} h^4 \|u_t\|_4 dt \right)^2 \\ &\leq C\tau^{-1} h^8 \int_0^{t_n} \|u_t\|_4^2 dt. \end{aligned} \quad (5.4.28)$$

Employ the Taylor expansion to get

$$\begin{aligned} \sum_{j=1}^n \|r_2^j\|_0^2 &\leq \sum_{j=1}^n \left(C\tau \int_{t_{j-1}}^{t_j} \|u_{ttt}\|_0 dt \right)^2 \\ &\leq C\tau^3 \int_0^{t_n} \|u_{ttt}\|_0^2 dt. \end{aligned} \quad (5.4.29)$$

It follows from (5.4.25)-(5.4.29) that

$$\begin{aligned} \|e^n\|_1 &\leq C \left\{ \|u_0 - u_{0h}\|_1 + h^3 \|u_0\|_4 + h^4 \left(\int_0^{t_n} \|u_t\|_4^2 dt \right)^{1/2} \right. \\ &\quad \left. + \tau^2 \left(\int_0^{t_n} \|u_{ttt}\|_0^2 dt \right)^{1/2} \right\}. \end{aligned} \quad (5.4.30)$$

This together with

$$\begin{aligned} &\|u(t_n) - P_h u(t_n)\|_1 \\ &\leq Ch^3 \|u(t_n)\|_4 \leq Ch^3 \left[\|u_0\|_4 + \int_0^{t_n} \|u_t\|_4 dt \right] \end{aligned}$$

yields (5.4.21). This completes the proof. $\square$

5.4.2 Quadratic element difference schemes for two-dimensional parabolic equations

Consider the following initial and boundary values problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f(x, y, t), & (x, y) \in \Omega, 0 < t \leq T, & (5.4.31a) \\ u(x, y, t) = 0, & (x, y) \in \partial\Omega, 0 < t \leq T, & (5.4.31b) \\ u(x, y, 0) = u_0(x, y), & (x, y) \in \Omega, & (5.4.31c) \end{cases}$$

where Ω is a planar polygon, and $f, u_0 \in L^2(\Omega)$. A corresponding variational problem is: Find $u = u(\cdot, t) \in H_0^1(\Omega)$ such that

$$\begin{cases} \left(\frac{\partial u}{\partial x}, v \right) + a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), 0 < t \leq T, & (5.4.32a) \\ (u(\cdot, 0), v) = (u_0, v), \quad \forall v \in H_0^1(\Omega), & (5.4.32b) \end{cases}$$

where

$$a(u, v) = \int_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx dy, \quad u, v \in H_0^1(\Omega).$$

As in §3.4, we place a triangulation $T_h = \{K_Q : Q \in \Omega_h^*\}$, and a corresponding dual grid $T_h^* = \{K_{P_0}^*, K_M^* : P_0 \in \bar{\Omega}_h, M \in \bar{M}_h\}$. (See §3.4 and Figg. 5.4.1 and 5.4.2 below for details and notations.)

The trial function space U_h is chosen as the Lagrange quadratic element space related to the triangulation T_h . The test space V_h is taken as the piecewise constant function space corresponding to the dual grid T_h^* , of which the basis functions are the characteristic functions ψ_{P_0} and ψ_M of $K_{P_0}^*$ and K_M^* , respectively.

The semi-discrete quadratic difference scheme reads: Find $u_h = u_h(\cdot, t) \in U_h$ ($0 < t \leq T$) such that

$$\begin{cases} \left(\frac{\partial u_h}{\partial x}, v_h \right) + a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, 0 < t \leq T, & (5.4.33a) \\ (u_h(\cdot, 0), v_h) = (u_0, v_h), \quad \forall v_h \in V_h, & (5.4.33b) \end{cases}$$

where $a(\cdot, \cdot)$ is interpreted in the sense of generalized functions. In particular, when v_h is taken as the basis functions ψ_{P_0} and ψ_M respectively, we have

$$a(u_h, \psi_{P_0}) = \int_{\partial K_{P_0}^*} \left(-\frac{\partial u_h}{\partial x} dy + \frac{\partial u_h}{\partial y} dx \right),$$

$$a(u_h, \psi_M) = \int_{\partial K_M^*} \left(-\frac{\partial u_h}{\partial x} dy + \frac{\partial u_h}{\partial y} dx \right).$$

The fully-discrete quadratic element difference scheme is: Find $u_h^n \in U_h$ ($n = 1, 2, \dots$) such that

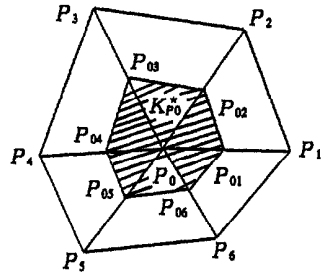


Fig. 5.4.1

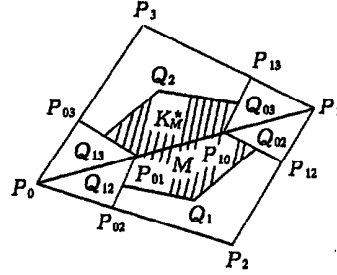


Fig. 5.4.2

$$\begin{cases} (\bar{\partial}_t u_h^n, v_h) + a(u_h^{n,\theta}, v_h) = (f^{n,\theta}, v_h), \\ \forall v_h \in V_h, n = 1, 2, \dots, \\ (u_h^0, v_h) = (u_0, v_h), \forall v_h \in V_h, \end{cases} \quad (5.4.34a)$$

$$(u_h^0, v_h) = (u_0, v_h), \forall v_h \in V_h, \quad (5.4.34b)$$

where (τ is the time step size and $t_n = n\tau$)

$$\bar{\partial}_t u_h^n = \frac{u_h^n - u_h^{n-1}}{\tau}, \quad u_h^{n,\theta} = \theta u_h^n + (1 - \theta) u_h^{n-1},$$

$$f^{n,\theta} = \theta f^n + (1 - \theta) f^{n-1}, \quad f^n = f(t_n).$$

(5.4.34) leads to a backward Euler fully-discrete scheme when $\theta = 1$, and a Crank-Nicolson fully-discrete scheme when $\theta = \frac{1}{2}$.

As in finite element methods, we can first compute the element mass matrices and the element stiff matrices, then pile them up to form the overall mass and stiff matrices respectively.

A direct computation gives

$$\left(\frac{\partial u_h}{\partial t}, v_h \right) = \sum_{K \in T_h} \{v_h\}_K^T B \{u_h\}_K,$$

where (cf. Fig. 5.4.3))

$$\{v_h\}_K^T = [v_h(P_i), v_h(P_j), v_h(P_k), v_h(M_i), v_h(M_j), v_h(M_k)],$$

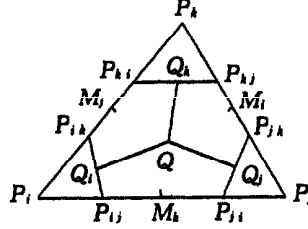


Fig. 5.4.3

$$\{u_h\}_K = \left[\frac{\partial u_h}{\partial t}(P_i), \frac{\partial u_h}{\partial t}(P_j), \frac{\partial u_h}{\partial t}(P_k), \right. \\ \left. \frac{\partial u_h}{\partial t}(M_i), \frac{\partial u_h}{\partial t}(M_j), \frac{\partial u_h}{\partial t}(M_k) \right]^T,$$

$$B = \frac{S_K}{1944} \begin{bmatrix} 96 & -16 & -16 & 8 & 72 & 72 \\ -16 & 96 & -16 & 72 & 8 & 72 \\ -16 & -16 & 96 & 72 & 72 & 8 \\ -38 & -13 & -13 & 300 & 98 & 98 \\ -13 & -38 & -13 & 98 & 300 & 98 \\ -13 & -13 & -38 & 98 & 98 & 300 \end{bmatrix}.$$

It is this B that is called the element mass matrix.

Also note that

$$a(u_h, v_h) = \sum_{K \in T_h} I_K(u_h, v_h),$$

$$I_K(u_h, v_h) = \{v_h\}_K^T A \{u_h\}_K,$$

where A is the element stiff matrix defined as follows:

$$A = \frac{1}{36S_K} [\bar{a}_{ij}]_{6 \times 6},$$

$$\begin{aligned} \bar{a}_{11} &= 10c^2, & \bar{a}_{12} &= a^2 - b^2 + c^2, \\ \bar{a}_{13} &= -a^2 + b^2 + c^2, & \bar{a}_{14} &= -4c^2, \end{aligned}$$

$$\begin{aligned}
\bar{a}_{15} &= 8a^2 - 8b^2 - 4c^2, & \bar{a}_{16} &= -8a^2 + 8b^2 - 4c^2, \\
\bar{a}_{21} &= a^2 - b^2 + c^2, & \bar{a}_{22} &= 10a^2, \\
\bar{a}_{23} &= a^2 + b^2 - c^2, & \bar{a}_{24} &= -4a^2 - 8b^2 + 8c^2, \\
\bar{a}_{25} &= -4a^2, & \bar{a}_{26} &= -4a^2 + 8b^2 - 8c^2, \\
\bar{a}_{31} &= -a^2 + b^2 + c^2, & \bar{a}_{32} &= a^2 + b^2 - c^2, \\
\bar{a}_{33} &= 10b^2, & \bar{a}_{34} &= -8a^2 - 4b^2 + 8c^2, \\
\bar{a}_{35} &= 8a^2 - 4b^2 - 8c^2, & \bar{a}_{36} &= -4b^2, \\
\bar{a}_{41} &= -2c^2, & \bar{a}_{42} &= -5a^2 - 3b^2 + 3c^2, \\
\bar{a}_{43} &= -3a^2 - 5b^2 + 3c^2, & \bar{a}_{44} &= 8a^2 + 8b^2 + 4c^2, \\
\bar{a}_{45} &= -8a^2 + 8b^2 - 4c^2, & \bar{a}_{46} &= 8a^2 - 8b^2 - 4c^2, \\
\bar{a}_{51} &= 3a^2 - 3b^2 - 5c^2, & \bar{a}_{52} &= -2a^2, \\
\bar{a}_{53} &= 3a^2 - 5b^2 - 3c^2, & \bar{a}_{54} &= -4a^2 + 8b^2 - 8c^2, \\
\bar{a}_{55} &= 4a^2 + 8b^2 + 8c^2, & \bar{a}_{56} &= -4a^2 - 8b^2 + 8c^2, \\
\bar{a}_{61} &= -3a^2 + 3b^2 - 5c^2, & \bar{a}_{62} &= -5a^2 + 3b^2 - 3c^2, \\
\bar{a}_{63} &= -2b^2, & \bar{a}_{64} &= 8a^2 - 4b^2 - 8c^2, \\
\bar{a}_{65} &= -8a^2 - 4b^2 + 8c^2, & \bar{a}_{66} &= 8a^2 + 4b^2 + 8c^2,
\end{aligned}$$

where $a = |\overline{P_i P_k}|$, $b = |\overline{P_i P_j}|$, and $c = |\overline{P_j P_k}|$.

The results of a numerical experiment are given in Table 5.4.1, where the Crank-Nicolson fully-discrete generalized difference scheme (5.4.34) (GDM) is compared with the linear finite element method (FEM1) and the quadratic finite element method (FEM2), for the following initial and boundary values problem:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, & (x, y) \in \Omega = (0, \pi) \times (0, \pi), \quad 0 < t \leq 1, \\ u|_{\partial\Omega} = 0, & 0 < t \leq 1, \\ u|_{t=0} = \sin x \cdot \sin y, & (x, y) \in \Omega. \end{cases}$$

Place a right angle triangulation with a space step size $\pi/4$ and a time step size $\tau = 0.001$. The average error and the maximum error

(in absolute values) of the approximate solutions, at all the nodes when $t = \frac{1}{2}$, and the true solution $u = e^{-2t} \sin x \cdot \sin y$ are given in Table 5.4.1.

Table 5.4.1 Comparison of approximation errors

	GDM	FEM1	FEM2
average error	0.005064	0.036336	0.000770
maximum error	0.009187	0.054020	0.001568

5.5 Generalized Difference Methods for Nonlinear Parabolic Equations

5.5.1 Problem and schemes

Let us consider the following initial and boundary values problem of nonlinear parabolic equations:

$$\begin{cases} \frac{\partial u}{\partial t} + Au = f(x, y, t), & (x, y) \in \Omega, 0 < t \leq T, & (5.5.1a) \\ u = 0, & (x, y) \in \partial\Omega, 0 < t \leq T, & (5.5.1b) \\ u = u_0(x, y), & (x, y) \in \Omega, t = 0, & (5.5.1c) \end{cases}$$

where

$$Au = -\nabla(a(x, y, u)\nabla u),$$

Ω is a planar polygonal region, u_0 a smooth function on $\bar{\Omega}$, f a smooth function on $\bar{\Omega} \times [0, T]$, and $a(x, y, u)$ a smooth function on $\bar{\Omega} \times \mathbb{R}$.

Place on Ω a triangulation T_h and its corresponding circumcenter dual grid T_h^* . Assume that any inner angle of each triangular element is not greater than $\frac{\pi}{2}$, and that T_h and T_h^* are quasi-uniform. Also assume the following "quasi-parallelogram condition" holds: there exists a constant $\mu > 0$ such that for any adjacent triangular elements K_Q and $K_{Q'}$ (cf. Fig. 5.5.1)

$$|\Delta_j^Q - \Delta_i^Q + \Delta_j^{Q'} - \Delta_i^{Q'}| \leq \mu h^3,$$

where, e.g., Δ_j^Q denotes the area of the triangle with vertexes Q, P_i and P_k .

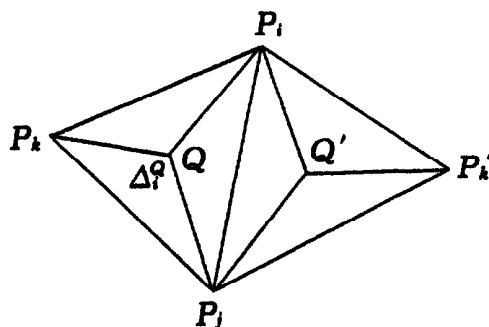


Fig. 5.5.1

Let U_h be the linear element space corresponding to T_h , and V_h the piecewise constant function space related to T_h^* . The semi-discrete generalized difference scheme approximating (5.5.1) is: Find $u_h = u_h(\cdot, t) \in U_h$ ($0 < t \leq T$) such that

$$\begin{cases} \left(\frac{\partial u_h}{\partial t}, v_h \right) + A(u_h; u_h, v_h) = (f, v_h), & (5.5.2a) \\ \forall v_h \in V_h, 0 < t \leq T, \\ u_h(x, y, 0) = u_{0h}(x, y), (x, y) \in \Omega, & (5.5.2b) \end{cases}$$

where

$$A(w; u, v) = \int_{\Omega} a(x, y, w) \nabla u \cdot \nabla v dx dy.$$

For $\bar{u}_h, u_h \in U_h$, let Π_h^* be the interpolation projection operator from U_h onto V_h , then we can write

$$\begin{aligned} & A(w; u_h, \Pi_h^* \bar{u}_h) \\ &= \sum_{K \in T_h} \sum_{l=i,j,k} |\overline{P_{l+1}P_{l+2}}| \int_{\overline{M_l Q}} a(x, y, w) \frac{\partial u_h}{\partial \tau_l} \frac{\partial \bar{u}_h}{\partial \tau_l} ds, \end{aligned} \quad (5.5.3)$$

where $\tau_l = \overline{P_{l+1}P_{l+2}}/|\overline{P_{l+1}P_{l+2}}|$ ($l = i, j, k$; $i+1 = j, j+1 = k, k+1 = i$). (cf. §4.5 and Fig. 5.5.2.) Assume u_{0h} is a certain approximation of u_0 in U_h , satisfying

$$\|u_0 - u_{0h}\|_0 \leq Ch.$$

The Crank-Nicolson fully-discrete generalized difference scheme for (5.5.1) is: Find $u_h^n \in U_h$ ($n = 1, 2, \dots, N$) such that

$$\begin{cases} (\bar{\partial}_t u_h^n, v_h) + A(u_h^{n-1/2}; u_h^{n-1/2}, v_h) = (f^{n-1/2}, v_h), & (5.5.4a) \\ \forall v_h \in V_h, n = 1, 2, \dots, N, \\ u_h^0 = u_{0h}, & (5.5.4b) \end{cases}$$

where τ is the time step size, $N = T/\tau$, $t_n = n\tau$ and

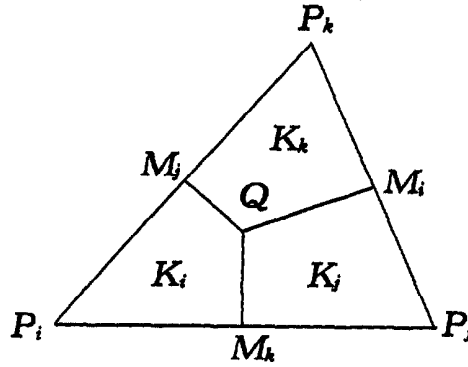


Fig. 5.5.2

$$\bar{\partial}_t u_h^n = \frac{u_h^n - u_h^{n-1}}{\tau}, \quad u_h^{n-1/2} = \frac{u_h^n + u_h^{n-1}}{2},$$

$$f^{n-1/2} = \frac{f^n + f^{n-1}}{2}, \quad f^n = f(x, y, t_n).$$

In the sequel we assume the following:

- (i) $a(x, y, u), \frac{\partial}{\partial t} a(x, y, u) \in C(\bar{\Omega} \times [0, T])$,
 $\forall (x, y) \in \Omega, u \in C(\bar{\Omega} \times [0, T])$.
- (ii) $0 < a_0 \leq a(x, y, u) \leq a_1 < +\infty, \left| \frac{\partial}{\partial t} a(x, y, u) \right| \leq a_1$,
 $\forall (x, y) \in \Omega, u \in C(\bar{\Omega} \times [0, T]), t \in [0, T]$.

- (iii) $|a(x, y, u) - a(x, y, w)| \leq M|u - w|,$
 $\forall (x, y) \in \Omega, u, v \in C(\bar{\Omega} \times [0, T]).$
- (iv) (5.5.1) has a unique solution u , and $u, u_t \in C([0, T]; C^2(\bar{\Omega}))$,
 $u_{tt} \in L^2((0, T); H^1(\Omega)), u_{ttt} \in L^2((0, T); L^2(\Omega)).$

Here and below, the following Banach function spaces are used. Let X be a Banach space equipped with a norm $\|\cdot\|_X$, m a nonnegative integer, $-\infty \leq a < b \leq \infty$, and $1 \leq p \leq \infty$. We define

$$C^m([a, b]; X) := \{u(t) : u(t) \text{ as a function from } [a, b] \text{ to } X \\ \text{is } m\text{-times continuously differentiable}\},$$

and

$$L^p((a, b); X) = \{u(t) : u(t) \in X, \forall t \in [a, b], \|u(t)\|_X \in L^p(a, b)\}.$$

In particular, we write

$$C([a, b]; X) = C^0([a, b]; X).$$

$C^m([a, b]; X)$ and $L^p((a, b); X)$ become Banach spaces when supplied with the following norms respectively:

$$\|u\|_{C^m([a, b]; X)} = \sum_{j=0}^m \max_{t \in [a, b]} \left\| \frac{\partial^j}{\partial t^j} u(t) \right\|_X,$$

$$\|u\|_{L^p((a, b); X)} = \begin{cases} \left\{ \int_a^b \|u(t)\|_X^p dt \right\}^{1/p}, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{t \in (a, b)} \|u(t)\|_X, & p = \infty. \end{cases}$$

Furthermore, if X is a Hilbert space with an inner product $(\cdot, \cdot)_X$, then $L^2((a, b); X)$ is a Hilbert space as well with the inner product

$$(u, v)_{L^2((a, b); X)} = \int_a^b (u(t), v(t))_X dt.$$

5.5.2 Some lemmas

In the error estimations later on, besides the results such as the equivalences of the norms given in §5.1, we also need some preliminary results presented below.

Lemma 5.5.1 *The following estimates holds for any $w \in C(\Omega \times [0, T])$*

$$A(w; u_h, \Pi_h^* \bar{u}_h) = A(w; \bar{u}_h, \Pi_h^* u_h), \quad \forall \bar{u}_h, u_h \in U_h, \quad (5.5.5)$$

$$A(w; u_h, \Pi_h^* u_h) \geq \beta \|u_h\|_1^2, \quad \forall u_h \in U_h, \quad (5.5.6)$$

$$|A(w; u_h, \Pi_h^* \bar{u}_h)| \leq C \|u_h\|_1 \|\bar{u}_h\|_1, \quad \forall \bar{u}_h, u_h \in U_h, \quad (5.5.7)$$

where β and C are positive constants independent of U_h .

The above results can be found in Lemma 5.1.2 and Theorem 4.5.1. The next lemma reveals some properties of $(u_h, \Pi_h^* \bar{u}_h)$ for the circumcenter decomposition, which are similar to those in Lemma 5.1.5 for the barycenter decomposition discussed there.

Lemma 5.5.2 *There exist positive constants C_0 and C such that*

$$(u_h, \Pi_h^* u_h) \geq C_0 \|u_h\|_0^2, \quad \forall u_h \in U_h, \quad (5.5.8)$$

$$|(u_h, \Pi_h^* \bar{u}_h)| \leq C \|u_h\|_0 \|\bar{u}_h\|_0, \quad \forall \bar{u}_h, u_h \in U_h, \quad (5.5.9)$$

$$|(u_h, \Pi_h^* \bar{u}_h) - (\bar{u}_h, \Pi_h^* u_h)| \leq Ch \|u_h\|_0 \|\bar{u}_h\|_0, \quad (5.5.10)$$

$$\forall \bar{u}_h, u_h \in U_h.$$

Proof Write $K = \triangle P_i P_j P_k \in T_h$, $K_l = K \cap K_{P_l}^*$, $u_l = u_h(P_l)$ ($l = i, j, k$), $u_Q = u_h(Q)$. (Cf. Fig. 5.5.2.) Since the inner angles of K are not greater than $\frac{\pi}{2}$, it is easy to verify that

$$\Delta_{l-1} + \Delta_{l+1} \geq \Delta_l, \quad (l = i, j, k) \quad (5.5.11)$$

and hence

$$\Delta_{l-1} + \Delta_{l+1} \geq \frac{1}{2} S_K, \quad (l = i, j, k) \quad (5.5.12)$$

where Δ_i , Δ_j and Δ_k denote the areas of $\triangle QP_jP_k$, $\triangle QP_kP_i$ and $\triangle QP_iP_j$, respectively.

Note that for any $u_h \in U_h$,

$$\begin{aligned} \int_{K_i} u_h dx dy &= \frac{1}{3} \left(u_i + \frac{1}{2}(u_i + u_j) + u_Q \right) \frac{\Delta_k}{2} \\ &\quad + \frac{1}{3} \left(u_i + \frac{1}{2}(u_i + u_k) + u_Q \right) \frac{\Delta_j}{2}, \end{aligned}$$

and

$$u_Q = (u_i \Delta_i + u_j \Delta_j + u_k \Delta_k) / S_K.$$

So we have

$$\begin{aligned} &\int_{K_i} u_h dx dy \\ &= \frac{1}{6} \left\{ u_i (\Delta_j + \Delta_k) \left(\frac{3}{2} + \frac{\Delta_i}{S_K} \right) + u_j \left[\frac{\Delta_k}{2} + (\Delta_j + \Delta_k) \frac{\Delta_j}{S_K} \right] \right. \\ &\quad \left. + u_k \left[\frac{\Delta_j}{2} + (\Delta_j + \Delta_k) \frac{\Delta_k}{S_K} \right] \right\}. \end{aligned} \tag{5.5.13}$$

This together with (5.5.11) and (5.5.12) gives

$$\begin{aligned} &\sum_{l=i,j,k} u_l \int_{K_l} u_h dx dy \\ &= \frac{1}{6} \sum_{l=i,j,k} \left\{ u_l^2 (\Delta_{l-1} + \Delta_{l+1}) \left(\frac{3}{2} + \frac{\Delta_l}{S_K} \right) \right. \\ &\quad + u_l u_{l+1} \left[\frac{\Delta_{l-1}}{2} + (\Delta_{l-1} + \Delta_{l+1}) \frac{\Delta_{l+1}}{S_K} \right] \\ &\quad \left. + u_l u_{l-1} \left[\frac{\Delta_{l+1}}{2} + (\Delta_{l-1} + \Delta_{l+1}) \frac{\Delta_{l-1}}{S_K} \right] \right\} \\ &= \frac{1}{6} \sum_{l=i,j,k} \left\{ u_l^2 (\Delta_{l-1} + \Delta_{l+1}) (\Delta_{l-1} + \Delta_{l+1} + 2\Delta_l) / S_K \right. \\ &\quad + u_l u_{l+1} (\Delta_{l-1} \Delta_{l+1} + \Delta_l \Delta_{l-1} + \Delta_l^2 + \Delta_{l+1}^2) / S_K \\ &\quad \left. + \frac{1}{2} (u_l + u_{l+1})^2 \Delta_{l-1} \right\} \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{6S_K} \sum_{l=i,j,k} [u_l^2(\Delta_{l-1}^2 + \Delta_{l+1}^2 + 2\Delta_l^2 + 2\Delta_{l-1}\Delta_{l+1}) \\
&\quad + u_l u_{l+1}(\Delta_l^2 + \Delta_{l+1}^2 + \Delta_{l-1}\Delta_{l+1} + \Delta_l\Delta_{l-1})] \\
&= \frac{1}{12S_K} \sum_{l=i,j,k} [u_l^2(\Delta_{l-1}^2 + \Delta_{l+1}^2 + 2\Delta_l^2 + 2\Delta_{l-1}\Delta_{l+1}) \\
&\quad + u_{l+1}^2(\Delta_l^2 + \Delta_{l-1}^2 + 2\Delta_{l+1}^2 + 2\Delta_l\Delta_{l-1}) \\
&\quad + 2u_l u_{l+1}(\Delta_l^2 + \Delta_{l+1}^2 + \Delta_{l-1}\Delta_{l+1} + \Delta_l\Delta_{l-1})] \\
&\geq \frac{1}{12S_K} \sum_{l=i,j,k} [(u_l + u_{l+1})^2(\Delta_l^2 + \Delta_{l+1}^2) \\
&\quad + (u_l\Delta_{l-1} + u_{l+1}\Delta_{l+1})^2 + (u_l\Delta_l + u_{l+1}\Delta_{l-1})^2] \tag{5.5.14} \\
&\geq \frac{1}{24S_K} \sum_{l=i,j,k} (u_l + u_{l+1})^2(\Delta_l + \Delta_{l+1})^2 \\
&\geq \frac{S_K}{96} \sum_{l=i,j,k} (u_l + u_{l+1})^2 \\
&= \frac{S_K}{96} \left[\sum_{l=i,j,k} u_l^2 + \left(\sum_{l=i,j,k} u_l \right)^2 \right] \\
&\geq \frac{S_K}{96} \sum_{l=i,j,k} u_l^2.
\end{aligned}$$

So by the equivalence of the norms there exists a constant $C_0 > 0$ such that

$$(u_h, \Pi_h^* u_h) = \sum_{K \in T_h} \sum_{l=i,j,k} u_l \int_{K_l} u_h dx dy \geq C_0 \|u_h\|_0^2, \quad \forall u_h \in U_h.$$

It is easy to see that

$$\begin{aligned}
(u_h, \Pi_h^* \bar{u}_h) &= \sum_{K \in T_h} \sum_{l=i,j,k} \bar{u}_l \int_{K_l} u_h dx dy \\
&\leq C \sum_{K \in T_h} \sum_{l,m=i,j,k} \bar{u}_l u_m S_K \\
&\leq C \|u_h\|_0 \|\bar{u}_h\|_0, \quad \forall \bar{u}_h, u_h \in U_h.
\end{aligned}$$

It follows from (5.5.13) that

$$\begin{aligned}
 & (u_h, \Pi_h^* w_h) - (w_h, \Pi_h^* u_h) \\
 &= \frac{1}{6S_K} \sum_{K \in T_h} \sum_{l=i,j,k} [w_l u_{l+1} (\Delta_{l+1} \Delta_{l-1} + \Delta_{l+1}^2 - \Delta_l^2 - \Delta_l \Delta_{l-1}) \\
 & \quad + w_{l+1} u_l (\Delta_l^2 + \Delta_l \Delta_{l-1} - \Delta_{l+1} \Delta_{l-1} - \Delta_{l+1}^2)] \\
 &= \frac{1}{6} \sum_{K \in T_h} \sum_{l=i,j,k} (w_l u_{l+1} - w_{l+1} u_l) (\Delta_{l+1} - \Delta_l).
 \end{aligned}$$

Summing the right-hand side over every side $L = \overline{P_i P_j}$, and using the boundary condition and the "quasi-parallelogram condition" yield the following estimate (cf. Fig. 5.5.1):

$$\begin{aligned}
 & |(u_h, \Pi_h^* w_h) - (w_h, \Pi_h^* u_h)| \\
 &= \left| \sum_L (w_i u_j - w_j u_i) (\Delta_j^Q - \Delta_i^Q + \Delta_j^{Q'} - \Delta_i^{Q'}) \right| \\
 &\leq Ch \left(\sum u_i^2 h^2 \right)^{1/2} \left(\sum w_i^2 h^2 \right)^{1/2} \\
 &\leq Ch \|u_h\|_0 \|w_h\|_0.
 \end{aligned}$$

This completes the proof. $\square$

Lemma 5.5.3 Assume $w \in C(\bar{\Omega} \times [0, T])$ and $u \in C([0, T]; H_0^1(\Omega) \cap C^2(\bar{\Omega}))$. Then there exists a constant $C > 0$ independent of the subspace U_h such that for all $\bar{u}_h, u_h \in U_h$

$$|A(w; u - u_h, \Pi_h^* \bar{u}_h)| \leq C(h + \|u - u_h\|_1) \|\bar{u}_h\|_1, \quad (5.5.15)$$

$$|A_t(w; u - u_h, \Pi_h^* \bar{u}_h)| \leq C(h + \|u - u_h\|_1) \|\bar{u}_h\|_1, \quad (5.5.16)$$

where

$$\begin{aligned}
 & A_t(w; v, \Pi_h^* \bar{u}_h) \\
 &= \sum_{K \in T_h} \sum_{l=i,j,k} \overline{P_{l+1} P_{l+2}} \int_{M_l Q} \frac{\partial a(x, y, w)}{\partial t} \frac{\partial v}{\partial \eta} \frac{\partial \bar{u}_h}{\partial \eta} ds.
 \end{aligned}$$

Proof It follows from (5.5.7) that

$$\begin{aligned}
& |A(w; u - u_h, \Pi_h^* \bar{u}_h)| \\
& \leq |A(w; u - \Pi_h u, \Pi_h^* \bar{u}_h)| + |A(w; \Pi_h u - u_h, \Pi_h^* \bar{u}_h)| \\
& \leq C |\nabla(u - \Pi_h u)|_\infty \sum_{K \in T_h} \sum_{l=i,j,k} \overline{P_{l+1} P_{l+2}} \int_{M_l Q} \left| \frac{\partial \bar{u}_h}{\partial \tau_l} \right| ds \quad (5.5.17) \\
& \quad + C \|\Pi_h u - u_h\|_1 \|\bar{u}_h\|_1.
\end{aligned}$$

Notice the following estimates

$$\begin{aligned}
& |\nabla(u - \Pi_h u)|_\infty \leq Ch \|u\|_{C([0,T];C^2(\bar{\Omega}))}, \\
& \sum_{K \in T_h} \sum_{l=i,j,k} \overline{P_{l+1} P_{l+2}} \int_{M_l Q} \left| \frac{\partial \bar{u}_h}{\partial \tau_l} \right| ds \\
& \leq C \sum_{K_Q \in T_h} \left(\left| \frac{\partial \bar{u}_h(Q)}{\partial x} \right| + \left| \frac{\partial \bar{u}_h(Q)}{\partial y} \right| \right) h^2 \\
& \leq C \left(\sum_{K_Q \in T_h} h^2 \right)^{1/2} \left(\sum_{K_Q \in T_h} \left(\left| \frac{\partial \bar{u}_h(Q)}{\partial x} \right|^2 + \left| \frac{\partial \bar{u}_h(Q)}{\partial y} \right|^2 \right) h^2 \right)^{1/2} \\
& \leq C \|\bar{u}_h\|_1. \\
& \|\Pi_h u - u_h\|_1 \leq \|\Pi_h u - u\|_1 + \|u - u_h\|_1 \\
& \leq Ch + \|u - u_h\|_1.
\end{aligned}$$

So (5.5.15) holds. (5.5.16) can be similarly proved. $\square$

Lemma 5.5.4 Let $P_h u$ be the elliptic projection of the solution $u = u(x, y, t)$ to (5.5.1), onto U_h , that is, let $P_h u \in U_h$ satisfy

$$A(u; P_h u - u, v_h) = 0, \quad \forall v_h \in V_h, \quad 0 < t \leq T. \quad (5.5.18)$$

Then there exists a constant $C > 0$ independent of U_h such that

$$\|u - P_h u\|_1 \leq Ch, \quad (5.5.19)$$

$$\left\| \frac{\partial}{\partial t} (u - P_h u) \right\|_1 \leq Ch. \quad (5.5.20)$$

Proof Since (5.5.18) is a linear system of equations, its solution $P_h u$ uniquely exists by (5.5.6). It follows from (5.5.6), (5.5.18) and (5.5.15) that

$$\begin{aligned} & \|\Pi_h u - P_h u\|_1^2 \\ & \leq CA(u; \Pi_h u - P_h u, \Pi_h^*(\Pi_h u - P_h u)) \\ & \leq C(h + \|\Pi_h u - u\|_1) \|\Pi_h u - P_h u\|_1. \end{aligned}$$

Hence

$$\|u - P_h u\|_1 \leq \|u - \Pi_h u\|_1 + \|\Pi_h u - P_h u\|_1 \leq Ch.$$

(5.5.20) can be proved in like manner by virtue of (5.5.16). This completes the proof. $\square$

Lemma 5.5.5 *If $u \in C([0, T]; H_0^1(\Omega) \cap C^2(\bar{\Omega}))$, then there exists a constant $C > 0$ independent of U_h such that*

$$\|\nabla P_h u\|_\infty \leq C, \quad 0 \leq t \leq T. \quad (5.5.21)$$

Proof It is obvious that

$$\begin{aligned} & \|\nabla P_h u\|_\infty \\ & \leq \|\nabla(P_h u - \Pi_h u)\|_\infty + \|\nabla(\Pi_h u - u)\|_\infty + \|\nabla u\|_\infty. \end{aligned}$$

By the inverse property of the finite element and (5.5.19) we have

$$\begin{aligned} & \|\nabla(P_h u - \Pi_h u)\|_\infty \leq Ch^{-1} \|P_h u - \Pi_h u\|_1 \\ & \leq Ch^{-1} (\|P_h u - u\|_1 + \|u - \Pi_h u\|_1) \leq C. \end{aligned}$$

Also note

$$\|\nabla(\Pi_h u - u)\|_\infty \leq Ch \|u\|_{2,\infty}.$$

Thus (5.5.21) holds. This completes the proof. $\square$

Lemma 5.5.6 *If $u \in C([0, T]; H_0^1(\Omega) \cap C^2(\bar{\Omega}))$, then there exists a constant $C > 0$ independent of U_h such that*

$$\begin{aligned} & |A(u; P_h u, \Pi_h^* \bar{u}_h) - A(w_h; P_h u, \Pi_h^* \bar{u}_h)| \\ & \leq C(h + \|u - w_h\|_0) \|\bar{u}_h\|_1, \quad \forall w_h, \bar{u}_h \in U_h. \end{aligned} \quad (5.5.22)$$

Proof The following estimate follows from (5.5.3), the hypothesis (iii), (5.5.21) and (5.5.18):

$$\begin{aligned} & |A(u; P_h u, \Pi_h^* \bar{u}_h) - A(w_h; P_h u, \Pi_h^* \bar{u}_h)| \\ &= \left| \sum_{K \in T_h} \sum_{l=i,j,k} |\bar{P}_{l+1} P_{l+2}| \right. \\ & \quad \cdot \left. \int_{M_l Q} [a(x, y, u) - a(x, y, w_h)] \frac{\partial P_h u}{\partial \tau_l} \frac{\partial \bar{u}_h}{\partial \tau_l} ds \right| \\ &\leq C \sum_{K \in T_h} \sum_{l=i,j,k} |\bar{P}_{l+1} P_{l+2}| \\ & \quad \cdot \int_{M_l Q} |u - w_h| \|\nabla P_h u\|_\infty \left| \frac{\partial \bar{u}_h}{\partial \tau_l} \right| ds \\ &\leq C \sum_{K \in T_h} \sum_{l=i,j,k} |\bar{P}_{l+1} P_{l+2}| \\ & \quad \cdot \int_{M_l Q} (|u - \Pi_h u| + |\Pi_h u - w_h|) \left| \frac{\partial \bar{u}_h}{\partial \tau_l} \right| ds \\ &\leq C(h|u|_2 \sum_{K \in T_h} \sum_{l=i,j,k} |\bar{P}_{l+1} P_{l+2}| \cdot \int_{M_l Q} \left| \frac{\partial \bar{u}_h}{\partial \tau_l} \right| ds \\ & \quad + \sum_{K \in T_h} \sum_{l=i,j,k} |\bar{P}_{l+1} P_{l+2}| \cdot \int_{M_l Q} |\Pi_h u - w_h| \left| \frac{\partial \bar{u}_h}{\partial \tau_l} \right| ds) \\ &\leq C(h|u|_2 \|\bar{u}_h\|_1 + \|\Pi_h u - w_h\|_0 \|\bar{u}_h\|_1) \\ &\leq C(h + \|u - w_h\|_0) \|\bar{u}_h\|_1. \end{aligned}$$

This completes the proof. $\square$

5.5.3 Error estimates

Theorem 5.5.1 *Let u and u_h be the solutions to the problem (5.5.1) and the semi-discrete generalized difference scheme (5.5.2) respectively, satisfying $u, u_t \in C([0, T]; C^2(\bar{\Omega}))$, then*

$$\max_{0 \leq t \leq T} \|u - u_h\|_0 + \left(\int_0^t \|u - u_h\|_1^2 dt \right)^{1/2} \leq Ch. \quad (5.5.23)$$

Proof It follows from (5.5.1), (5.5.2) and (5.5.19) that

$$\begin{aligned} & (e_t, v_h) + A(u_h; e, v_h) \\ = & -(\rho_t, v_h) + A(u_h; P_h u, v_h) - A(u; P_h u, v_h), \end{aligned} \quad (5.5.24)$$

where

$$e = P_h u - u_h, \quad \rho = u - P_h u.$$

Setting $v_h = \Pi_h^* e$ and using (5.5.10) and (5.5.22) yield

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (e, \Pi_h^* e) + A(u_h; e, \Pi_h^* e) \\ = & \frac{1}{2} [(e, \Pi_h^* e_t) - (e_t, \Pi_h^* e)] - (\rho_t, \Pi_h^* e) \\ & + A(u_h; P_h u, \Pi_h^* e) - A(u; P_h u, \Pi_h^* e) \\ \leq & Ch \|e\|_0 \|e_t\|_0 + C \|\rho_t\|_0 \|e\|_0 \\ & + C(h + \|u - u_h\|_0) \|e\|_1. \end{aligned} \quad (5.5.25)$$

If we set $v_h = \Pi_h^* e_t$ in (5.5.24) and employ (5.5.8), (5.5.7), (5.5.9) and (5.5.22), then we have

$$\begin{aligned} & \|e_t\|_0^2 \\ \leq & C(\|e\|_1 \|e_t\|_1 + \|\rho_t\|_0 \|e_t\|_0 + (h + \|u - u_h\|_0) \|e_t\|_1) \\ \leq & C(h^{-1} \|e\|_1 + \|\rho_t\|_0 + h^{-1}(h + \|u - u_h\|_0)) \|e_t\|_0. \end{aligned}$$

Thus

$$h \|e_t\|_0 \leq C(\|e\|_1 + \|\rho_t\|_0 + h + \|\rho\|_0 + \|e\|_0). \quad (5.5.26)$$

Inserting (5.5.26) into (5.5.25) gives

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (e, \Pi_h^* e) + A(u_h; e, \Pi_h^* e) \\ & \leq C \|e\|_0 (\|e\|_1 + \|\rho_t\|_0 + h + \|\rho\|_0) \\ & \quad + C \|e\|_1 (h + \|\rho\|_0 + \|e\|_0). \end{aligned}$$

This together with (5.5.20) and (5.5.21) implies

$$\frac{1}{2} \frac{d}{dt} (e, \Pi_h^* e) + A(u_h; e, \Pi_h^* e) \leq C (\|e\|_0 + h) \|e\|_1.$$

Integrate it on t and use (5.5.8) and (5.5.6), then we have

$$\begin{aligned} & \|e\|_0^2 + \int_0^t \|e\|_1^2 dt \\ & \leq C (\|e(0)\|_0^2 + \int_0^t (\|e\|_0 + h) \|e\|_1 dt) \\ & \leq C (\|e(0)\|_0^2 + \int_0^t (C(\|e\|_0^2 + h^2) + \frac{1}{2C} \|e\|_1^2) dt). \end{aligned}$$

This results in

$$\|e\|_0^2 + \int_0^t \|e\|_1^2 dt \leq C (h^2 + \int_0^t \|e\|_0^2 dt).$$

So by the Gronwall's inequality we have

$$\|e\|_0^2 + \int_0^t \|e\|_1^2 dt \leq C h^2.$$

Finally, this together with (5.5.20) and (5.5.21) leads to the desired result (5.5.23). This completes the proof. $\square$

Theorem 5.5.2 *Let u and u_h^n be the solutions to the problem (5.5.1) and the Crank-Nicolson fully-discrete generalized difference scheme (5.5.4) respectively, satisfying $u, u_t \in C([0, T]; C^2(\bar{\Omega}))$, $u_{tt} \in L^2((0, T); H^2(\Omega))$, and $u_{ttt} \in L^2((0, T); L^2(\Omega))$. Then we have*

$$\begin{aligned} & \max_{1 \leq n \leq N} \left\{ \|u_h^n - u(t_n)\|_0^2 + \tau \sum_{i=1}^n \|u_h^{i-1/2} - u(t_{i-1/2})\|_1^2 \right\} \\ & \leq C(h^2 + \tau^4). \end{aligned}$$

The proof to this theorem is omitted to save the space (cf. [A-52]).

Bibliography and Comments

The development of the theory of generalized difference methods for parabolic equations is parallel to that for elliptic equations. Generalized difference methods for parabolic equations are proposed and discussed in [B-57]. [A-53] considers a Hermite type cubic element difference scheme for a one-dimensional parabolic equation (cf. §5.4). Discussed respectively in [A-20,21,43] are the generalized difference method and its variety—a mass concentration method for two-dimensional parabolic equations. [A-52,23,46] deal with the generalized difference methods for nonlinear parabolic equations. [A-6] is concerned with a quadratic element generalized difference scheme for a heat-transfer equation (§5.4). The extreme value property and the uniform convergence is studied in [A-58].

In some early references on generalized difference methods for parabolic equations, the proofs to the error estimates are not quite rigorous, due to a wrong presumption that the L^2 -estimate (the dual argument) still holds for linear element generalized difference methods for elliptic equations. As regards the error estimates of semi- and fully-discrete generalized difference methods, we can borrow the theories and techniques of finite element methods to get basically parallel results. But there are certain difficulties requiring special treatments, such as the asymmetry of $(\cdot, \Pi_h^* \cdot)$. A method dealing with the asymmetry of $(\cdot, \Pi_h^* \cdot)$ is given in §5.5.

Problem 1 Discuss the error estimates for high order element difference schemes for two-dimensional parabolic equations.

Problem 2 Consider the generalized difference method for $u_{tt} = \Delta^2 u$ (cf. the first four sections of Chapter 4), of which the one-dimensional case has been discussed in [A-42].

Chapter 6

HYPERBOLIC EQUATIONS

Hyperbolic equations, especially first order hyperbolic systems, have important applications in fluid mechanics and light propagation. Due to the special properties of this class of equations, the difference method remains to be the most used method to solve them. In this chapter, we introduce an extension of the classical difference method, i.e., the generalized difference method, in particular the upwind generalized difference method.

6.1 Generalized Difference Methods for Second Order Hyperbolic Equations

Consider the mixed problem of second order hyperbolic equations:

$$\begin{cases} u_{tt} + Au = f(x, t), & x \in \Omega, 0 < t \leq T, & (6.1.1a) \\ u = 0, & x \in \partial\Omega, 0 < t \leq T, & (6.1.1b) \\ u = u_0(x), u_t = u_1(x), & x \in \Omega, t = 0, & (6.1.1c) \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded region with a piecewise smooth boundary $\partial\Omega$; $u_{tt} = \frac{\partial^2 u}{\partial t^2}$; A is a uniformly elliptic second order partial differential operator:

$$Au = - \sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right),$$

where $a_{ij}(x) = a_{ji}(x)$ are sufficiently smooth. By the uniform ellipticity we mean the existence of a constant $\alpha > 0$ satisfying

$$a(u, u) \equiv \int_{\Omega} \left(\sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) dx \geq \alpha |u|_1^2, \quad \forall u \in H_0^1(\Omega). \quad (6.1.2)$$

A variational form of (6.1.1) is: Find $u(\cdot, t) \in H_0^1(\Omega)$ ($0 < t \leq T$) such that

$$\begin{cases} (u_{tt}, v) + a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \quad 0 < t \leq T, & (6.1.3a) \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, & (6.1.3b) \end{cases}$$

where $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ inner product,

$$a(u, v) = \int_{\Omega} \left(\sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right) dx, \quad (6.1.4)$$

and the solution to (6.1.3) is referred to as a generalized solution of (6.1.1).

6.1.1 Semi-discrete generalized difference scheme

For simplicity, let Ω be a planar convex polygonal region. As in the previous chapters, we place a quasi-uniform triangulation T_h and a barycenter dual grid T_h^* on Ω , and accordingly construct a piecewise linear trial function space $U_h \subset H_0^1(\Omega)$ and a piecewise constant test function space $V_h \subset L^2(\Omega)$. Then, the semi-discrete scheme reads: Find $u_h = u_h(\cdot, t) \in U_h$ such that

$$\begin{cases} (u_{htt}, v_h) + a(u_h, v_h) = (f, v_h), & \forall v_h \in V_h, & (6.1.5a) \\ u_h(x, 0) = u_{0h}(x), \quad u_{ht}(x, 0) = u_{1h}(x), & x \in \Omega, & (6.1.5b) \end{cases}$$

where $a(\cdot, \cdot)$ is the bilinear form defined by (6.1.4). But on $U_h \times V_h$, this is only a formal definition and calls for further explanations: It is obtained by integrating (Au, v) in parts either on individual dual elements, or on the whole Ω in the sense of generalized functions (cf.

§3.1). u_{0h} and u_{1h} are certain approximations of $u_0(x)$ and $u_1(x)$ respectively, usually taken as their interpolation projections or L^2 -projections into U_h . The latter is equivalent to replace (6.1.5b) by

$$\begin{cases} (u_h(\cdot, 0), v_h) = (u_0, v_h), \\ (u_{ht}(\cdot, 0), v_h) = (u_1, v_h), \end{cases} \quad \forall v_h \in V_h.$$

Let $\{\phi_j(x)\}_{j=1,2,\dots,n}$ and $\{\psi_j(x)\}_{j=1,2,\dots,n}$ be bases of U_h and V_h respectively. Then we can state (6.1.5) in the following fashion: Find an approximate solution in the form

$$u_h = \sum_{j=1}^n \mu_j(t) \phi_j(x)$$

such that its coefficients $\mu_1(t), \dots, \mu_n(t)$ solve the following initial value problem of ordinary differential equations:

$$\begin{cases} \sum_{j=1}^n \left[\frac{d^2 \mu_j(t)}{dt^2} (\phi_j, \psi_i) + \mu_j(t) a(\phi_j, \psi_i) \right] = (f, \psi_i), & (6.1.5a)' \\ \mu_i(0) = \alpha_i, \quad \mu_{it}(0) = \beta_i, & (6.1.5b)' \end{cases}$$

where $0 < t \leq T$, $i = 1, 2, \dots, n$, α_i 's are the coefficients in $u_{0h} = \sum_{i=1}^n \alpha_i \phi_i$, and β_i 's are the coefficients in $u_{1h} = \sum_{i=1}^n \beta_i \phi_i$. It is easy to check that the matrix $M = \{(\phi_j, \psi_i)\}$ is symmetric and positive definite, so (6.1.5)' admits a unique and smooth solution for each $f \in L^2(\Omega)$.

Now we deduce the H^1 -estimate of the error $u - u_h$ in a way similar to that in §5.1. First let us define an elliptic projection operator $P_h: H^2(\Omega) \cap H_0^1(\Omega) \rightarrow U_h$ in terms of the following generalized difference equation:

$$a(P_h w, v_h) = a(w, v_h), \quad \forall v_h \in V_h. \quad (6.1.6)$$

Recalling the results in §3.2 we have

$$\|w - P_h w\|_1 \leq Ch |w|_2. \quad (6.1.7)$$

Now let u and u_h be the solutions to (6.1.3) and (6.1.5) respectively. Then the error $(u - u_h)$ satisfies

$$(u_{tt} - u_{htt}, v_h) + a(u - u_h, v_h) = 0, \quad \forall v_h \in V_h. \quad (6.1.8)$$

Set

$$\rho = u - P_h u, \quad e = P_h u - u_h, \quad (6.1.9)$$

then $u - u_h = \rho + e$. We shall need the following estimates for ρ and e (see (6.1.7)):

$$\begin{aligned} \|\rho\|_1 &= \|u - P_h u\|_1 \leq Ch\|u\|_2 \\ &\leq Ch\left(\|u_0\|_2 + \int_0^t \|u_t\|_2 dt\right), \end{aligned} \quad (6.1.10a)$$

$$\|\rho_{tt}\|_0 = \|P_h u_{tt} - u_{tt}\|_0 \leq Ch\|u_{tt}\|_2, \quad (6.1.10b)$$

$$\begin{aligned} \|e(0)\|_1 &= \|P_h u_0 - u_{0h}\|_1 \\ &\leq \|P_h u_0 - u_0\|_1 + \|u_0 - u_{0h}\|_1 \\ &\leq Ch\|u_0\|_2 + \|u_0 - u_{0h}\|_1, \end{aligned} \quad (6.1.10c)$$

$$\begin{aligned} \|e_t(0)\|_0 &\leq \|P_h u_1 - u_1\|_1 + \|u_1 - u_{1h}\|_0 \\ &\leq Ch\|u_1\|_2 + \|u_1 - u_{1h}\|_0. \end{aligned} \quad (6.1.10d)$$

Rewrite (6.1.8) into

$$(e_{tt}, v_h) + (\rho_{tt}, v_h) + a(e, v_h) + a(\rho, v_h) = 0.$$

This together with (6.1.6) gives

$$(e_{tt}, v_h) + a(e, v_h) = -(\rho_{tt}, v_h), \quad \forall v_h \in V_h. \quad (6.1.11)$$

As in §5.1, let us introduce the interpolation projection operator $\Pi_h^* : H_0^1(\Omega) \rightarrow V_h$ and set $v_h = \Pi_h^* e_t$ in (6.1.11) to obtain

$$(e_{tt}, \Pi_h^* e_t) + a(e, \Pi_h^* e_t) = -(\rho_{tt}, \Pi_h^* e_t). \quad (6.1.12)$$

As in §5.1 we have

$$(u_h, \Pi_h^* \bar{u}_h) = (\bar{u}_h, \Pi_h^* u_h), \quad \forall \bar{u}_h, u_h \in U_h,$$

$$(u_h, \Pi_h^* u_h) > 0, \quad \forall u_h \neq 0.$$

Write $|||u_h|||_0 = (u_h, \Pi_h^* u_h)^{1/2}$, then the norms $|||u_h|||_0$ and $\|u_h\|_0$ are equivalent. Moreover, by the inverse property the following estimate holds:

$$|a(u_h, \Pi_h^* \bar{u}_h) - a(\bar{u}_h, \Pi_h^* u_h)| \leq Ch\|u_h\|_1\|\bar{u}_h\|_1 \leq C\|u_h\|_0\|\bar{u}_h\|_1,$$

$$\forall \bar{u}_h, u_h \in U_h. \quad (6.1.13)$$

(6.1.12) is equivalent to

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|e_t\|_0^2 + \frac{1}{2} \frac{d}{dt} a(e, \Pi_h^* e) \\ &= \frac{1}{2} [a(e_t, \Pi_h^* e) - a(e, \Pi_h^* e_t)] - (\rho_{tt}, \Pi_h^* e_t). \end{aligned}$$

It follows from (6.1.13) and $\|\Pi_h^* e_t\|_0 \leq C \|e_t\|_0$ that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|e_t\|_0^2 + \frac{1}{2} \frac{d}{dt} a(e, \Pi_h^* e) \\ & \leq C \|e_t\|_0 \|e\|_1 + \|\rho_{tt}\|_0 \|\Pi_h^* e_t\|_0 \\ & \leq C [\|e_t\|_0^2 + \|e\|_1^2 + \|\rho_{tt}\|_0^2]. \end{aligned}$$

Integrate it to obtain

$$\begin{aligned} & \|e_t\|_0^2 + a(e, \Pi_h^* e) \\ & \leq \|e_t(0)\|_0^2 + a(e(0), \Pi_h^* e(0)) \\ & \quad + C \int_0^t [\|e_t\|_0^2 + \|e\|_1^2 + \|\rho_{tt}\|_0^2] dt. \end{aligned} \quad (6.1.14)$$

Note

$$\begin{aligned} a(e, \Pi_h^* e) & \geq \alpha \|e\|_1^2, \quad \alpha > 0 \text{ a constant,} \\ a(e(0), \Pi_h^* e(0)) & \leq C \|e(0)\|_1^2, \\ \|e_t(0)\|_0^2 & \leq C \|e_t(0)\|_0^2. \end{aligned}$$

Therefore, by (6.1.10) and (6.1.14) we have

$$\begin{aligned} & \|e_t\|_0^2 + \|e\|_1^2 \\ & \leq C \left\{ \|P_h u_0 - u_{0h}\|_1^2 + \|P_h u_1 - u_{1h}\|_0^2 + h^2 \int_0^t \|u_{tt}\|_2^2 dt \right. \\ & \quad \left. + \int_0^t (\|e_t\|_0^2 + \|e\|_1^2) dt \right\}. \end{aligned}$$

So it follows from the Gronwall's inequality that

$$\|e\|_1^2 \leq C \left\{ \|P_h u_0 - u_{0h}\|_1^2 + \|P_h u_1 - u_{1h}\|_0^2 + h^2 \int_0^t \|u_{tt}\|_2^2 dt \right\}.$$

Combining this with (6.1.10a) yields the error estimate:

$$\begin{aligned} \|u - u_h\|_1^2 &\leq C \left\{ \|P_h u_0 - u_{0h}\|_1^2 + \|P_h u_1 - u_{1h}\|_0^2 \right. \\ &\quad \left. + h^2 \left[\|u_0\|_2^2 + \int_0^t \|u_t\|_2^2 dt + \int_0^t \|u_{tt}\|_2^2 dt \right] \right\}. \end{aligned} \quad (6.1.15)$$

6.1.2 Fully-discrete generalized difference scheme

Now we further discretize time t of the semi-discrete difference scheme (6.1.5) to deduce fully-discrete schemes. Let the time step size be τ and $t_n = n\tau$ ($n = 0, 1, \dots, N$; $N\tau = T$), $u_h^n = u_h(t_n)$. For a function v well-defined at times $t = t_n$ ($n = 0, 1, \dots, N$), we shall use the following symbols:

$$\begin{aligned} v^n &= v|_{t=t_n}, \quad v^{n+1/2} = \frac{v^n + v^{n+1}}{2}, \quad \hat{\partial}_t v^{n+1/2} = \frac{v^{n+1} - v^n}{\tau}, \\ v^{n,1/4} &= \frac{1}{4}(v^{n+1} + 2v^n + v^{n-1}) = \frac{1}{2}(v^{n+1/2} + v^{n-1/2}), \\ \hat{\partial}_t v^n &= \frac{v^{n+1} - v^{n-1}}{2\tau} = \frac{1}{\tau}(v^{n+1/2} - v^{n-1/2}) = \frac{1}{2}(\hat{\partial}_t v^{n+1/2} + \hat{\partial}_t v^{n-1/2}), \\ \partial_{tt} v^n &= \frac{v^{n+1} - 2v^n + v^{n-1}}{\tau^2} = \frac{1}{\tau}(\hat{\partial}_t v^{n+1/2} - \hat{\partial}_t v^{n-1/2}). \end{aligned}$$

Now, we use weighted averages of the values of u_h and f at t_{n-1} , t_n and t_{n+1} to construct the following fully-discrete generalized difference scheme:

$$(\partial_{tt} u_h^n, v_h) + a(u_h^{n,1/4}, v_h) = (f^{n,1/4}, v_h), \quad \forall v_h \in V_h. \quad (6.1.16)$$

This is an implicit scheme, being absolutely stable as shown below. For readers familiar with finite difference methods, it is not difficult to recall the counterpart of (6.1.16) in finite difference methods. (cf. [A-27].)

Let us deduce the convergence estimate. Assume u is a smooth solution of the continuous problem (6.1.3). By the Taylor expansion we have

$$u_{tt}^{n,1/4} = \partial_{tt} u^n - r_n, \quad (6.1.17a)$$

where the remainder r_n satisfies the following estimate (cf. [B-27])

$$\|r_n\|_0^2 \leq C\tau^3 \int_{t_{n-1}}^{t_{n+1}} \left\| \frac{\partial^4 u}{\partial t^4} \right\|_0^2 dt. \quad (6.1.17b)$$

So by (6.1.3a) we have

$$(\partial_{tt}u^n - r_n, v) + a(u^{n,1/4}, v) = (f^{n,1/4}, v),$$

which gives by setting $v = v_h$ that

$$(\partial_{tt}u^n, v_h) + a(u^{n,1/4}, v_h) = (r_n, v_h) + (f^{n,1/4}, v_h).$$

Subtracting it with (6.1.16) yields the error equation

$$(\partial_{tt}(u^n - u_h^n), v_h) + a(u^{n,1/4} - u_h^{n,1/4}, v_h) = (r_n, v_h). \quad (6.1.18)$$

Set

$$u^n - u_h^n = (u^n - P_h u^n) + (P_h u^n - u_h^n) = \rho^n + e^n.$$

Then by (6.1.6) and (6.1.18) we have

$$(\partial_{tt}e^n, v_h) + a(e^{n,1/4}, v_h) = (r_n - \partial_{tt}\rho^n, v_h).$$

If in particular we choose $v_h = \Pi_h^* \hat{\partial}_t e^n$, then

$$(\partial_{tt}e^n, \Pi_h^* \hat{\partial}_t e^n) + a(e^{n,1/4}, \Pi_h^* \hat{\partial}_t e^n) = (r_n - \partial_{tt}\rho^n, \Pi_h^* \hat{\partial}_t e^n). \quad (6.1.19)$$

Let us deal with respectively these terms in the above equality. For the first term on the left-hand side we have

$$\begin{aligned} & (\partial_{tt}e^n, \Pi_h^* \hat{\partial}_t e^n) \\ &= \frac{1}{2\tau} ((e^{n+1} - 2e^n + e^{n-1})\tau^{-1}, \Pi_h^* (e^{n+1} - e^{n-1})\tau^{-1}) \\ &= \frac{1}{2\tau} (\hat{\partial}_t e^{n+1/2} - \hat{\partial}_t e^{n-1/2}, \Pi_h^* (\hat{\partial}_t e^{n+1/2} + \hat{\partial}_t e^{n-1/2})) \\ &= \frac{1}{2\tau} [(\hat{\partial}_t e^{n+1/2}, \Pi_h^* \hat{\partial}_t e^{n+1/2}) - (\hat{\partial}_t e^{n-1/2}, \Pi_h^* \hat{\partial}_t e^{n-1/2})] \\ &= \frac{1}{2\tau} [|||\hat{\partial}_t e^{n+1/2}|||_0^2 - |||\hat{\partial}_t e^{n-1/2}|||_0^2]. \end{aligned}$$

For the second term on the left-hand side of (6.1.19) we have

$$\begin{aligned}
 & a(e^{n,1/4}, \Pi_h^* \hat{\partial}_t e^n) \\
 &= \frac{1}{2\tau} [a(e^{n+1/2}, \Pi_h^* e^{n+1/2}) - a(e^{n-1/2}, \Pi_h^* e^{n-1/2})] \\
 & \quad - \frac{1}{2\tau} [a(e^{n+1/2}, \Pi_h^* e^{n-1/2}) - a(e^{n-1/2}, \Pi_h^* e^{n+1/2})].
 \end{aligned}$$

It follows from (6.1.13) that

$$\begin{aligned}
 & \frac{1}{2\tau} |a(e^{n+1/2}, \Pi_h^* e^{n-1/2}) - a(e^{n-1/2}, \Pi_h^* e^{n+1/2})| \\
 &= \frac{1}{2\tau} |a(e^{n+1/2} - e^{n-1/2}, \Pi_h^* e^{n-1/2}) \\
 & \quad - a(e^{n-1/2}, \Pi_h^* (e^{n+1/2} - e^{n-1/2}))| \\
 &= \frac{1}{2} |a(\hat{\partial}_t e^n, \Pi_h^* e^{n-1/2}) - a(e^{n-1/2}, \Pi_h^* \hat{\partial}_t e^n)| \\
 &\leq C \|e^{n-1/2}\|_1 \|\hat{\partial}_t e^n\|_0 \\
 &\leq C (\|e^{n-1/2}\|_1^2 + \|\hat{\partial}_t e^{n+1/2}\|_0^2 + \|\hat{\partial}_t e^{n-1/2}\|_0^2).
 \end{aligned}$$

For the right-hand side of (6.1.19) we have

$$\begin{aligned}
 & |(r_n - \partial_{tt} \rho^n, \Pi_h^* \hat{\partial}_t e^n)| \\
 &\leq \|r_n\|_0^2 + \|\partial_{tt} \rho^n\|_0^2 + \frac{1}{2} \|\Pi_h^* \hat{\partial}_t e^{n+1/2}\|_0^2 + \frac{1}{2} \|\Pi_h^* \hat{\partial}_t e^{n-1/2}\|_0^2.
 \end{aligned}$$

Hence, (6.1.19) results in

$$\begin{aligned}
 & \frac{1}{2\tau} [|\|\hat{\partial}_t e^{n+1/2}\|_0^2 - |\|\hat{\partial}_t e^{n-1/2}\|_0^2|] \\
 & \quad + \frac{1}{2\tau} [a(e^{n+1/2}, \Pi_h^* e^{n+1/2}) - a(e^{n-1/2}, \Pi_h^* e^{n-1/2})] \\
 &\leq C \{ \|e^{n-1/2}\|_1^2 + \|\hat{\partial}_t e^{n+1/2}\|_0^2 + \|\hat{\partial}_t e^{n-1/2}\|_0^2 + \|r_n\|_0^2 + \|\partial_{tt} \rho^n\|_0^2 \\
 & \quad + \|\Pi_h^* \hat{\partial}_t e^{n+1/2}\|_0^2 + \|\Pi_h^* \hat{\partial}_t e^{n-1/2}\|_0^2 \}.
 \end{aligned}$$

Multiply it by 2τ , and sum it over $n = 1, 2, \dots, N-1$ to obtain

$$\begin{aligned}
 & |||\hat{\partial}_t e^{N-1/2}|||_0^2 + a(e^{N-1/2}, \Pi_h^* e^{N-1/2}) \\
 & \leq |||\hat{\partial}_t e^{1/2}|||_0^2 + a(e^{1/2}, \Pi_h^* e^{1/2}) \\
 & \quad + C\tau \sum_{n=1}^{N-1} [||e^{n-1/2}||_1^2 + \|\hat{\partial}_t e^{n+1/2}\|_0^2 + \|\hat{\partial}_t e^{n-1/2}\|_0^2 \\
 & \quad + \|r_n\|_0^2 + \|\partial_{tt}\rho^n\|_0^2 + \|\Pi_h^* \hat{\partial}_t e^{n+1/2}\|_0^2 + \|\Pi_h^* \hat{\partial}_t e^{n-1/2}\|_0^2].
 \end{aligned} \tag{6.1.20}$$

Notice (6.1.17b) and

$$\begin{aligned}
 & a(e^{n-1/2}, \Pi_h^* e^{n-1/2}) \geq \alpha \|e^{n-1/2}\|_1^2, \\
 & \sum_{n=1}^{N-1} \|\partial_{tt}\rho^n\|_0^2 = \frac{1}{\tau^2} \sum_{n=1}^{N-1} \left\| \int_{t^n}^{t^{n+1}} \int_{t-\tau}^t \rho_{tt}(s) ds dt \right\|_0^2 \\
 & \leq \frac{1}{\tau} \sum_{n=1}^{N-1} \int_{t^{n-1}}^{t^{n+1}} \|\rho_{tt}\|_0^2 dt \leq \frac{2}{\tau} \int_0^T \|\rho_{tt}\|_0^2 dt \leq \frac{Ch^2}{\tau} \int_0^T \|u_{tt}\|_2^2 dt.
 \end{aligned}$$

Also note the equivalence of the norms $|||\cdot|||_0$ and $\|\cdot\|_0$ on U_h . Then, (6.1.20) leads to

$$\begin{aligned}
 & \|\hat{\partial}_t e^{N-1/2}\|_0^2 + \|e^{N-1/2}\|_1^2 \\
 & \leq C \left\{ \|\hat{\partial}_t e^{1/2}\|_0^2 + \|e^{1/2}\|_1^2 + \tau^4 \int_0^T \|u_{tttt}\|_0^2 dt \right. \\
 & \quad \left. + h^2 \int_0^T \|u_{tt}\|_2^2 dt + \tau \sum_{n=1}^N (\|\hat{\partial}_t e^{n-1/2}\|_0^2 + \|e^{n-1/2}\|_1^2) \right\}.
 \end{aligned}$$

Finally, by virtue of Gronwall's theorem we have

$$\begin{aligned}
 & \|\hat{\partial}_t e^{N-1/2}\|_0^2 + \|e^{N-1/2}\|_1^2 \\
 & \leq C \left\{ \|\hat{\partial}_t e^{1/2}\|_0^2 + \|e^{1/2}\|_1^2 + \tau^4 \int_0^T \|u_{tttt}\|_0^2 dt \right. \\
 & \quad \left. + h^2 \int_0^T \|u_{tt}\|_2^2 dt \right\}.
 \end{aligned}$$

This together with (6.1.10) validates the following error estimate for the fully-discrete scheme.

Theorem 6.1.1 *Let u and u_h^n be the solutions to (6.1.9) and (6.1.16) respectively. Then the following error estimate holds:*

$$\begin{aligned}
& \|u(t_{n+1/2}) - u_h^{n+1/2}\|_1^2 \\
& \leq C \left\{ \|(P_h u - u_h)^{1/2}\|_1^2 + \|\hat{\partial}_t(P_h u - u_h)^{1/2}\|_0^2 + \tau^4 \int_0^T \|u_{tttt}\|_0^2 dt \right. \\
& \quad \left. + h^2 \left(\|u_0\|_2^2 + \int_0^T \|u_t\|_2^2 dt + \int_0^T \|u_{tt}\|_2^2 dt \right) \right\}.
\end{aligned} \tag{6.1.21}$$

Remark For the first term in the right-hand side of the inequality (6.1.21), we have

$$\begin{aligned}
& (P_h u - u_h)^{1/2} \\
& = \frac{1}{2}(P_h u^0 - u_h^0) + \frac{1}{2}(P_h u^1 - u_h^1) \\
& = \frac{1}{2}(P_h u_0 - u_{0h}) + \frac{1}{2}P_h(u^1 - u^0) + \frac{1}{2}P_h(u^0 - u_h^1) \\
& = \frac{1}{2}P_h(u_0 - u_{0h}) + \frac{\tau}{2}P_h(u^1 - u^0)/\tau - \frac{\tau}{2}P_h(u_h^1 - u^0)/\tau.
\end{aligned}$$

The accuracy of the first and third terms above is determined by the choices of u_{0h} and u_h^1 , and the second term is of order $O(\tau)$ thanks to the smoothness of the solution u .

6.2 Generalized Upwind Schemes for First Order Hyperbolic Equations

The classical upwind scheme occupies a very important position in the approximation of first order hyperbolic equations, due to its nice stability and monotonicity. But this scheme has only first order accuracy and suits solely rectangular grids. In this and the next sections, we construct a class of accurate generalized upwind schemes on irregular networks, including the classical upwind scheme as a special case.

6.2.1 Generalized Upwind Schemes

Let $\Omega \subset \mathbb{R}^2$ be a polygonal region with boundary $\partial\Omega$. Corresponding to a vector function $a = a(x) \in \mathbb{R}^2$, $x = (x_1, x_2) \in \bar{\Omega}$, we divide $\partial\Omega$ into two parts

$$\begin{cases} (\partial\Omega)_- = \{x \in \partial\Omega : a \cdot \nu \leq 0\} & \text{(flow in)} \end{cases} \quad (6.2.1a)$$

$$\begin{cases} (\partial\Omega)_+ = \{x \in \partial\Omega : a \cdot \nu > 0\} & \text{(flow out)} \end{cases} \quad (6.2.1b)$$

where ν stands for the unit outer normal vector of $\partial\Omega$.

Consider a mixed problem of first order partial differential equations:

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + a \cdot \nabla u(x, t) + \sigma(x, t)u(x, t) = f(x, t), \\ (x, t) \in \Omega \times [0, T], \end{cases} \quad (6.2.2a)$$

$$u(x, t) = 0, \quad (x, t) \in (\partial\Omega)_- \times [0, T], \quad (6.2.2b)$$

$$u(x, 0) = \phi(x), \quad x \in \bar{\Omega}, \quad (6.2.2c)$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$; σ , f and ϕ are smooth (scalar) functions; and a is a vector function. If these functions are sufficiently smooth, the problem (6.2.2) has a unique and smooth solution (See [B-53]).

As in §3.2, we place a quasi-uniform triangulation $T_h = \{K\}$ of Ω . Choose T_h^* to be the barycenter dual grid relative to T_h . Let P_0 be a node of T_h (cf. Fig. 3.2.1) with neighbouring nodes P_i ($1 \leq i \leq 6$), M_i the midpoint of $\overline{P_0 P_i}$, Q_i the circumcenter of $\triangle P_0 P_i P_{i+1}$ ($1 \leq i \leq 6$ and $P_7 = P_1$), and $K_{P_0}^*$ the dual element surrounding P_0 .

Recalling that $\mathcal{P}_r$ is the polynomial family of degree r , let us construct a finite element space

$$V_h = \{v_h : v_h|_{K^*} \in \mathcal{P}_r, \forall K^* \subset T_h^*; v_h = 0, \text{ on } K_{P_0}^* \text{ for } P_0 \in (\partial\Omega)_-\}.$$

Its basis functions for an interior node $P_0 = (x_1^{(0)}, x_2^{(0)})$ of T_h are taken as

$$v_{P_0}(x) = \begin{cases} \frac{1}{l!(m-l)!} (x_1 - x_1^{(0)})^l (x_2 - x_2^{(0)})^{m-l}, & x \in K_{P_0}^*, \\ 0, & \text{elsewhere,} \end{cases} \quad (6.2.3)$$

$$0 \leq l \leq m, 0 \leq m \leq r.$$

Due to the discontinuity of V_h on the boundaries of the dual elements, one can not apply the Galerkin finite element method on the entire region Ω . But it is feasible to apply it on a single dual element $K_{P_0}^*$. So we seek $u_h(\cdot, t) \in V_h$ satisfying

$$\int_{K_{P_0}^*} \left[\frac{\partial u_h}{\partial t} + a \cdot \nabla u_h + \sigma u_h \right] v_h dx = \int_{K_{P_0}^*} f v_h dx, \quad v_h \in V_h. \quad (6.2.4)$$

Denote by ν the unit outer normal vector of $\partial K_{P_0}^*$, and employ Green's formula

$$\begin{aligned} & \int_{K_{P_0}^*} (a \cdot \nabla u_h) v_h dx \\ &= - \int_{K_{P_0}^*} u_h \operatorname{div}(a v_h) dx + \int_{\partial K_{P_0}^*} (a \cdot \nu) u_h v_h ds, \end{aligned} \quad (6.2.5)$$

then we can rewrite (6.2.4) as

$$\begin{aligned} & \int_{K_{P_0}^*} \frac{\partial u_h}{\partial t} v_h dx - \int_{K_{P_0}^*} u_h \operatorname{div}(a v_h) dx \\ &+ \int_{K_{P_0}^*} \sigma u_h v_h dx + \int_{\partial K_{P_0}^*} (a \cdot \nu) u_h v_h ds \\ &= \int_{K_{P_0}^*} f v_h dx, \quad v_h \in V_h. \end{aligned} \quad (6.2.6)$$

Similarly as in (6.2.1), we can define $(\partial K_{P_0}^*)_-$ and $(\partial K_{P_0}^*)_+$. For $x \in \partial K_{P_0}^*$, set

$$\begin{aligned} u_h^+(x) &= \begin{cases} \lim_{\substack{x' \rightarrow x \\ x' \notin K_{P_0}^*}} u_h(x'), & \text{when } x \in (\partial K_{P_0}^*)_-, \\ \lim_{\substack{x' \rightarrow x \\ x' \in K_{P_0}^*}} u_h(x'), & \text{when } x \in (\partial K_{P_0}^*)_+, \end{cases} \\ u_h^-(x) &= \begin{cases} \lim_{\substack{x' \rightarrow x \\ x' \in K_{P_0}^*}} u_h(x'), & \text{when } x \in (\partial K_{P_0}^*)_-, \\ \lim_{\substack{x' \rightarrow x \\ x' \notin K_{P_0}^*}} u_h(x'), & \text{when } x \in (\partial K_{P_0}^*)_+. \end{cases} \end{aligned}$$

They are referred to as the upwind and the downwind values of $u_h(x)$ at $x \in \partial K_{P_0}^*$, respectively. On the analogy of the classical upwind scheme, we replace $u_h(x)$ in the line integral of the left-hand side of (6.2.6) by u_h^+ to obtain

$$\begin{aligned} & \int_{K_{P_0}^*} \frac{\partial u_h}{\partial t} v_h dx - \int_{K_{P_0}^*} u_h \operatorname{div}(av_h) dx + \int_{K_{P_0}^*} \sigma u_h v_h dx \\ & + \int_{\partial K_{P_0}^*} (a \cdot \nu) u_h^+ v_h ds = \int_{K_{P_0}^*} f v_h dx. \end{aligned} \quad (6.2.7)$$

It follows from (6.2.5) that

$$\begin{aligned} & - \int_{K_{P_0}^*} u_h \operatorname{div}(av_h) dx \\ & = \int_{K_{P_0}^*} (a \cdot \nabla u_h) v_h dx - \int_{\partial K_{P_0}^*} (a \cdot \nu) u_h v_h ds \\ & = \int_{K_{P_0}^*} (a \cdot \nabla u_h) v_h dx - \int_{(\partial K_{P_0}^*)_+} (a \cdot \nu) u_h^+ v_h ds \\ & \quad - \int_{(\partial K_{P_0}^*)_-} (a \cdot \nu) u_h^- v_h ds. \end{aligned}$$

Substituting it in (6.2.7) yields a semi-discrete upwind scheme:

$$\begin{aligned} & \int_{K_{P_0}^*} \frac{\partial u_h}{\partial t} v_h dx + \int_{K_{P_0}^*} (a \cdot \nabla u_h) v_h dx \\ & + \int_{K_{P_0}^*} \sigma u_h v_h dx + \int_{(\partial K_{P_0}^*)_-} (a \cdot \nu) [u_h] v_h ds \\ & = \int_{K_{P_0}^*} f v_h dx, \quad v_h \in V_h, \end{aligned} \quad (6.2.8a)$$

where $[u_h] = u_h^+ - u_h^-$ is the jump of u_h across $(\partial K_{P_0}^*)_-$. The initial and boundary value conditions are

$$\begin{cases} u_h(x, t) = 0, & x \in (\partial \Omega)_-, \end{cases} \quad (6.2.8b)$$

$$\begin{cases} u_h(x, 0) = \phi_h(x), & x \in \Omega, \end{cases} \quad (6.2.8c)$$

where $\phi_h(x)$ is a certain approximation of $\phi(x)$.

Equation (6.2.8a) can also be expressed in a symmetric form:

$$\begin{aligned} & \int_{K_{P_0}^*} \frac{\partial u_h}{\partial t} v_h dx + \int_{K_{P_0}^*} (a \cdot \nabla u_h) v_h dx \\ & + \int_{K_{P_0}^*} \sigma u_h v_h dx - \frac{1}{2} \int_{\partial K_{P_0}^*} (|a \cdot \nu| - a \cdot \nu) [u_h] v_h ds \\ & = \int_{K_{P_0}^*} f v_h dx, \quad v_h \in V_h. \end{aligned} \quad (6.2.9)$$

Various kinds of finite difference quotients can be used to further discretize the time derivative $\frac{\partial u_h}{\partial t}$, such as forward difference, backward difference, or Crank-Nicolson difference. It will be illustrated in Section 3 below that our scheme here leads to a classical upwind scheme if the space dimension is one and V_h consists of step functions. If on the other hand, V_h consists of piecewise high degree (> 1) polynomials, then the convergence rate of the approximate solutions increase accordingly, resulting in highly accurate upwind schemes.

6.2.2 Semi-discrete error estimate

In the equation (6.2.2a), we may assume without loss of generality that $\bar{\sigma} = \sigma - \frac{1}{2} \operatorname{div} a \geq \sigma_0 > 0$. In fact, otherwise we only have to perform the transformation

$$\bar{u} = u e^{-\omega t},$$

$$\omega = \sigma_0 + \sup_{(x,t) \in \Omega \times [0,T]} |\sigma(x,t)| + \frac{1}{2} \sup_{x \in \Omega} |\operatorname{div} a(x)|$$

to validate this assumption. Now, we define a bilinear form

$$\begin{aligned} a(u, v) = & \sum_{P_0} \left[\int_{K_{P_0}^*} (a \cdot \nabla u) v dx \right. \\ & \left. + \int_{(\partial K_{P_0}^*)_-} (a \cdot \nu) [u] v ds + \int_{K_{P_0}^*} \sigma u v dx \right], \end{aligned} \quad (6.2.10)$$

where $\sum_{P_0}$ denotes the sum over all the interior nodes P_0 of T_h and the boundary nodes on $(\partial\Omega)_+$. If the solution of (6.2.10) $u = u(\cdot, t) \in C^2((0, T); H^{r+1}(\Omega))$ for some $r > 0$, then the imbedding theorem guarantees that the jump of u across the inner boundaries $[u] = u^+ - u^- = 0$. Therefore, we may write (6.2.10) in a equivalent form:

$$(u_t, v) + a(u, v) = (f, v), \quad \forall v \in L^2(\Omega). \quad (6.2.11)$$

In terms of $a(u, v)$ the semi-discrete scheme (6.2.8a) can be written as

$$\sum_{P_0} \int_{K_{P_0}^*} \frac{\partial u_h}{\partial t} v_h dx + a(u_h, v_h) = \sum_{P_0} \int_{K_{P_0}^*} f v_h dx, \quad \forall v_h \in V_h. \quad (6.2.12)$$

By means of Green's formula and

$$\operatorname{div}(av_h) = v_h \operatorname{div} a + a \cdot \nabla v_h,$$

we have

$$\int_{\partial K_{P_0}^*} (a \cdot \nu) v_h^2 ds = 2 \int_{K_{P_0}^*} v_h (a \cdot \nabla v_h) dx + \int_{K_{P_0}^*} v_h^2 \operatorname{div} a dx.$$

Hence

$$\begin{aligned} & a(v_h, v_h) \\ &= \sum_{P_0} \left[\frac{1}{2} \int_{\partial K_{P_0}^*} (a \cdot \nu) v_h^2 ds - \frac{1}{2} \int_{K_{P_0}^*} v_h^2 \operatorname{div} a dx \right. \\ & \quad \left. + \int_{(\partial K_{P_0}^*)_-} (a \cdot \nu) [v_h] v_h ds + \int_{K_{P_0}^*} \sigma v_h^2 dx \right] \\ &= \sum_{P_0} \left[\frac{1}{2} \int_{(\partial K_{P_0}^*)_+} (a \cdot \nu) v_h^2 ds + \frac{1}{2} \int_{(\partial K_{P_0}^*)_-} (a \cdot \nu) v_h^2 ds \right. \\ & \quad \left. + \int_{(\partial K_{P_0}^*)_-} (a \cdot \nu) (v_h^+ - v_h^-) v_h ds \right. \\ & \quad \left. + \frac{1}{2} \int_{(\partial K_{P_0}^*)_-} (a \cdot \nu) [(v_h^+)^2 - 2v_h^+ v_h^- + (v_h^-)^2] ds \right. \\ & \quad \left. - \frac{1}{2} \int_{(\partial K_{P_0}^*)_-} (a \cdot \nu) [v_h]^2 ds + \int_{K_{P_0}^*} \bar{\sigma} v_h^2 dx \right]. \end{aligned}$$

Notice

$$v_h|_{(\partial K_{P_0}^*)_+} = v_h^+|_{\partial K_{P_0}^*}, \quad v_h|_{(\partial K_{P_0}^*)_-} = v_h^-|_{\partial K_{P_0}^*}.$$

Thus,

$$\begin{aligned} & a(v_h, v_h) \\ &= \frac{1}{2} \sum_{P_0} \left[\int_{(\partial K_{P_0}^*)_-} (a \cdot \nu) (v_h^+)^2 ds + \int_{(\partial K_{P_0}^*)_+} (a \cdot \nu) (v_h^+)^2 ds \right. \\ & \quad \left. - \int_{(\partial K_{P_0}^*)_-} (a \cdot \nu) [v_h]^2 ds + 2 \int_{K_{P_0}^*} \bar{\sigma} v_h^2 dx \right]. \end{aligned} \quad (6.2.13)$$

On the common side of $K_{P_0}^*$ and $K_{P_1}^*$

$$(\partial K_{P_0}^*)_+ = (\partial K_{P_1}^*)_-, \quad (\partial K_{P_0}^*)_- = (\partial K_{P_1}^*)_+.$$

Hence the first and second terms on the right-hand side of (6.2.13) cancel out each other on the inner boundaries of the elements, resulting in

$$\begin{aligned} & a(v_h, v_h) \\ &= -\frac{1}{2} \int_{(\partial\Omega)_-} |a \cdot \nu| (v_h^+)^2 ds + \frac{1}{2} \int_{(\partial\Omega)_+} |a \cdot \nu| (v_h^+)^2 ds \\ & \quad + \frac{1}{2} \sum_{P_0} \int_{(\partial K_{P_0}^*)_-} |a \cdot \nu| [v_h]^2 ds + \sum_{P_0} \int_{K_{P_0}^*} \bar{\sigma} v_h^2 dx. \end{aligned} \quad (6.2.14)$$

But $v_h^+|_{(\partial\Omega)_-} = 0$, so $a(v_h, v_h)$ is positive definite:

$$a(v_h, v_h) \geq \gamma_0 (\|v_h\|_0^2 + \|v_h\|_{\partial\Omega}^2), \quad (6.2.15)$$

where $\gamma_0 = \min(\sigma_0, \frac{1}{2})$, $\|v_h\|_0^2 = (v_h, v_h)$, and

$$\|v_h\|_{\partial\Omega}^2 = \sum_{P_0} \int_{(\partial K_{P_0}^*)_-} |a \cdot \nu| [v_h]^2 ds + \int_{(\partial\Omega)_+} |a \cdot \nu| (v_h^+)^2 ds.$$

It is an easy matter to show the stability of the semi-discrete scheme (6.2.8) by means of the positive definiteness of $a(v_h, v_h)$ (cf.

[B-60]). Now, let us define for $u \in H^1(\Omega)$ the Ritz projection $R_h u \in V_h$ determined by the equation

$$a(R_h u, v_h) = a(u, v_h), \quad \forall v_h \in V_h. \quad (6.2.16)$$

Since $a(v_h, v_h)$ is positive definite, the Ritz projection exists and is unique. Moreover, if $u(\cdot, t) \in H^{r+1}(\Omega)$ ($r \geq 0$), then there holds the following estimates (cf. [B-60]):

$$\|u - R_h u\|_0^2 + \|u - R_h u\|_{\partial\Omega}^2 \leq Ch \|u\|_1^2, \quad \text{when } r = 0, \quad (6.2.17)$$

$$|||u - R_h u||| \leq Ch^{r+1/2} \|u\|_{r+1}, \quad \text{when } r \geq 1, \quad (6.2.18)$$

where $|||\cdot|||$ is defined by

$$|||v|||^2 = \|v\|_0^2 + \|v\|_{\partial\Omega}^2 + h \sum_{P_0} \int_{K_{P_0}^*} (a \cdot \nabla v)^2 dx. \quad (6.2.19)$$

Using the arguments in §5.1 we can prove the following error estimate for the semi-discrete solution $u_h(t)$ (cf. [B-60]):

$$\begin{aligned} & \|u - u_h\|_0^2 + \int_0^t \|u - u_h\|_{\partial\Omega}^2 dt \\ & \leq C \left\{ \|\phi - \phi_h\|_0^2 + h^{2r+1} \left[\|\phi\|_{r+1}^2 + \|u\|_{r+1}^2 + \int_0^t \|u_t\|_{r+1}^2 dt \right] \right\}, \end{aligned} \quad (6.2.20)$$

where $\|v\|_{r+1}$ stands for the $H^{r+1}(\Omega)$ norm of $v(\cdot, t)$.

6.2.3 Fully-discrete error estimates

For sake of simplicity, we assume $\sigma(x, t) = 0$. So we consider the hyperbolic equation:

$$\frac{\partial u}{\partial t} + a(x) \cdot \nabla u = f(x, t), \quad (x, t) \in \Omega \times (0, T],$$

subject to the initial and boundary conditions (6.2.1b) and (6.2.1c). The semi-discrete upwind scheme is (6.2.12), but now

$$\begin{aligned} & a(u_h, v_h) \\ & = \sum_{P_0} \left[\int_{K_{P_0}^*} (a \cdot \nabla u_h) v_h dx + \int_{(\partial K_{P_0}^*)_-} (a \cdot \nu) [u_h] v_h ds \right]. \end{aligned} \quad (6.2.21)$$

Take a time step size $\tau > 0$, and write $u_h^n = u_h(x, t_n)$, $f^n = f(x, t_n)$, $t_n = n\tau$. Using the backward differencing on the time direction yields a backward differencing implicit scheme:

$$\int_{\Omega} u_h^n v_h dx + \tau a(u_h^n, v_h) = \int_{\Omega} (u_h^{n-1} + \tau f^n) v_h dx, \quad \forall v_h \in V_h. \quad (6.2.22)$$

Theorem 6.2.1 *Let u and u_h^n be the solutions to (6.2.1) and (6.2.22) respectively, satisfying $u_t \in H^{r+1}(\Omega)$, $u_{tt} \in L^2(\Omega)$, $u(x, 0) = \phi(x) \in H^{r+1}(\Omega)$, and $u_h^0(x) = \phi_h(x) \in V_h$. Then there holds the following error estimate:*

$$\begin{aligned} & \|u(t_n) - u_h^n\|_0 \\ \leq & \|\phi - \phi_h\|_0 + Ch^{r+1/2} \|\phi\|_{r+1} + \tau \int_0^{t_n} \|u_{tt}(t)\|_0 dt \\ & + Ch^{r+1/2} \int_0^{t_n} \|u_t(t)\|_{r+1} dt. \end{aligned} \quad (6.2.23)$$

Proof Note $u_h^n - u(t_n) = \rho^n + e^n$, where

$$\rho^n = R_h u(t_n) - u(t_n), \quad e^n = u_h^n - R_h u(t_n).$$

It follows from (6.2.18) that

$$\|\rho^n\| \leq Ch^{r+1/2} \|u(t_n)\|_{r+1}.$$

Also observe that

$$u(t_n) = u(0) + \int_0^{t_n} u_t(t) dt,$$

$$\|u(t_n)\|_{r+1} \leq \|u(0)\|_{r+1} + \int_0^{t_n} \|u_t(t)\|_{r+1} dt.$$

Thus

$$\|\rho^n\|_0 \leq Ch^{r+1/2} \left(\|\phi\|_{r+1} + \int_0^{t_n} \|u_t(t)\|_{r+1} dt \right). \quad (6.2.24)$$

Next, we turn to deal with e^n . Write $\bar{\partial}_t u_h^n = (u_h^n - u_h^{n-1})/\tau$. It follows from (6.2.22) and the definition of R_h that

$$\begin{aligned} & a(e^n, v_h) \\ &= a(u_h^n - R_h u(t_n), v_h) \\ &= -(\bar{\partial}_t u_h^n, v_h) + (f^n, v_h) - a(u(t_n), v_h) \\ &= (u_t(t_n) - \bar{\partial}_t u_h^n, v_h). \end{aligned}$$

Hence

$$\begin{aligned} & (\bar{\partial}_t e^n, e^n) + a(e^n, e^n) \\ &= (u_t(t_n) - R_h \bar{\partial}_t u(t_n), e^n) \\ &= (w_1^n + w_2^n, e^n), \end{aligned} \quad (6.2.25)$$

where

$$w_1^n = u_t(t_n) - \bar{\partial}_t u(t_n), \quad w_2^n = \bar{\partial}_t u(t_n) - R_h \bar{\partial}_t u(t_n).$$

Notice

$$(e^n)^+|_{(\partial\Omega)_-} = 0, \quad a(e^n, e^n) \geq 0.$$

So by (6.2.25) we have

$$\begin{aligned} \|e^n\|_0 &\leq \|e^{n-1}\|_0 + \tau \|w_1^n + w_2^n\|_0 \\ &\leq \|e^0\|_0 + \tau \sum_{j=1}^n \|w_1^j + w_2^j\|_0. \end{aligned} \quad (6.2.26)$$

It is obvious that

$$\begin{aligned} \|e^0\|_0 &= \|u_h^0 - R_h u(0)\|_0 \\ &\leq \|u_h^0 - u(x, 0)\|_0 + \|u(x, 0) - R_h u(x, 0)\|_0 \\ &\leq \|\phi - \phi_h\|_0 + Ch^{r+1/2} \|\phi\|_{r+1}. \end{aligned} \quad (6.2.27)$$

Also note

$$\begin{aligned} w_1^j &= u_t(t_j) - \tau^{-1}(u(t_j) - u(t_{j-1})) \\ &= \tau^{-1} \int_{t_{j-1}}^{t_j} (t - t_j) u_{tt}(t) dt, \end{aligned}$$

$$\begin{aligned}
w_2^j &= (I - R_h) \bar{\partial}_t u(t_j) \\
&= \tau^{-1} \int_{t_{j-1}}^{t_j} (I - R_h) u_t(t) dt.
\end{aligned}$$

So we have

$$\tau \sum_{j=1}^n \|w_1^j + w_2^j\|_0 \leq \tau \int_0^{t_n} \|u_{tt}(t)\|_0 dt + Ch^{r+1/2} \int_0^{t_n} \|u_t(t)\|_{r+1} dt. \quad (6.2.28)$$

Inserting (6.2.27) and (6.2.28) into (6.2.26) yields

$$\begin{aligned}
\|e^n\|_0 &\leq \|\phi - \phi_h\|_0 + Ch^{r+1/2} \|\phi\|_{r+1} + \tau \int_0^{t_n} \|u_{tt}(t)\|_0 dt \\
&\quad + Ch^{r+1/2} \int_0^{t_n} \|u_t(t)\|_{r+1} dt.
\end{aligned} \quad (6.2.29)$$

Finally, a combination of (6.2.24) and (6.2.29) leads to the desired estimate

$$\begin{aligned}
&\|u_h^n - u(t_n)\|_0 \\
&\leq \|e^n\|_0 + \|\rho^n\|_0 \leq \|\phi - \phi_h\|_0 + Ch^{r+1/2} \|\phi\|_{r+1} \\
&\quad + \tau \int_0^{t_n} \|u_{tt}(t)\|_0 dt + Ch^{r+1/2} \int_0^{t_n} \|u_t(t)\|_{r+1} dt.
\end{aligned} \quad (6.2.30)$$

This completes the proof. $\square$

If we approximate the derivative in (6.2.12) by a weighted differencing, then we have the following six-point difference scheme:

$$\begin{aligned}
&(u_h^n, v_h) + \tau a(\theta u_h^n + (1 - \theta) u_h^{n-1}, v_h) \\
&= (u_h^{n-1} + \tau \theta f^n + \tau(1 - \theta) f^{n-1}, v_h), \quad \forall v_h \in V_h,
\end{aligned} \quad (6.2.31)$$

where $\theta \in [0, 1]$ is a parameter. In particular, choosing $\theta = \frac{1}{2}$ results in a Crank-Nicolson scheme:

$$\begin{aligned}
&(u_h^n, v_h) + \tau a((u_h^n + u_h^{n-1})/2, v_h) \\
&= (u_h^{n-1} + \tau(f^n + f^{n-1})/2, v_h), \quad \forall v_h \in V_h.
\end{aligned} \quad (6.2.32)$$

Theorem 6.2.2 Assume the following: $\frac{1}{2} \leq \theta \leq 1$; $u(x, t)$ and $u_h^n(x)$ are the solutions to (6.2.1) and (6.2.31) respectively; $u_t \in H^{r+1}(\Omega)$; $u_{tt} \in L^2(\Omega)$; $u(x, 0) = \phi(x) \in H^{r+1}(\Omega)$; and $u_h^0(x) = \phi_h(x) \in V_h$. Then (6.2.30) holds for the approximate solution u_h^n .

The proof of this theorem is analogous to that of Theorem 6.2.1, and is left for interested readers.

6.3 Generalized Upwind Schemes for First Order Hyperbolic Systems

In this section we extend the generalized upwind difference scheme to solve positive symmetric hyperbolic systems.

6.3.1 Integral forms

Let $\Omega \subset R^2$ be a bounded region with a piecewise smooth boundary $\partial\Omega$, and $u(x, t) \in R^m$ ($x \in \bar{\Omega}$, $0 \leq t \leq T$). Consider the first order positive symmetric hyperbolic system:

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + A(x) \cdot \nabla u(x, t) + K(x, t)u(x, t) = f(x, t), \\ x = (x_1, x_2) \in \Omega, 0 < t \leq T, \end{cases} \quad (6.3.1a)$$

$$(B - M)u(x, t) = 0, \quad x \in \partial\Omega, \quad (6.3.1b)$$

$$u(x, 0) = \phi(x), \quad x \in \Omega, \quad (6.3.1c)$$

where $K(x, t)$ is an $m \times m$ coefficient matrix; $A = (A_1, A_2)$; A_i ($i = 1, 2$) are $m \times m$ real symmetric matrices; $B = \sum_{i=1}^2 n_i A_i$; $n = (n_1, n_2)$ is the unit outer normal vector of the boundary $\partial\Omega$; $M = M(x, t)$ is an $m \times m$ matrix; $f(x, t)$ and $\phi(x)$ are m -dimensional vector function. All these matrices and vectors are given. We also assume the following conditions:

$$M + M^T \geq 0, \quad \text{on } \partial\Omega, \quad (6.3.2a)$$

$$K + K^T - \sum_{i=1}^2 \frac{\partial A_i}{\partial x_i} \geq \sigma_0 I, \quad \text{on } \Omega, \quad (6.3.2b)$$

$$\ker(B - M) + \ker(B + M) = R^m, \text{ on } \partial\Omega, \quad (6.3.2c)$$

where $\sigma_0 > 0$ is a constant, and

$$\ker E = \{v \in R^m : Ev = 0\}. \quad (6.3.3)$$

Under the above conditions plus certain smoothness assumptions, problem (6.3.1) possesses a unique solution (cf. [B-31]).

Define an operator L by

$$Lu = \sum_{i=1}^2 A_i(x) \frac{\partial u}{\partial x_i} + K(x, t)u, \quad x \in \Omega, \quad (6.3.4)$$

and its formal conjugate by

$$L^*u = - \sum_{i=1}^2 \frac{\partial}{\partial x_i} (A_i(x)u) + K(x, t)^T u. \quad (6.3.5)$$

For $u, v \in (H^1(\Omega))^m$, we may use Green's formula to get an extended Green's formula

$$(Lu, v)_\Omega = (u, L^*v)_\Omega + (Bu, v)_{\partial\Omega}. \quad (6.3.6)$$

Here and below we adopt the symbols

$$(u, v)_\Omega = \int_\Omega \langle u, v \rangle dx, \quad (u, v)_{\partial\Omega} = \int_{\partial\Omega} \langle u, v \rangle ds.$$

Here the symbol $\langle u, v \rangle$ denotes the R^m inner product of u, v . In particular,

$$(Lv, v)_\Omega = \frac{1}{2}((L + L^*)v, v)_\Omega + \frac{1}{2}(Bv, v)_{\partial\Omega}. \quad (6.3.7)$$

In terms of (6.3.6) we can write (6.3.1a) in an integral form:

$$\left(\frac{\partial u}{\partial t}, v\right)_\Omega - (u, L^*v)_\Omega + (Bu, v)_{\partial\Omega} = (f, v)_\Omega, \quad (6.3.8)$$

6.3.2 Generalized upwind difference schemes

As in §6.2, we assume that Ω is a polygonal region, that $T_h = \{K\}$ is a triangulation of Ω , and $T_h^* = \{K_{P_0}^*\}$ is a barycenter dual grid. The finite element space V_h is defined as in §6.2, namely,

$$V_h = \{v_h : v_h|_{K^*} \in \mathcal{P}_r, \forall K^* \in T_h^*; v_h = 0 \text{ on } K_{P_0}^* \text{ for } P_0 \in (\partial\Omega)_-\}.$$

Since we are dealing with vector functions, we introduce $\bar{V}_h = [V_h]^m = V_h \times \cdots \times V_h$ (an m -multiplicative space). Employ (6.3.8) on each $K_{P_0}^*$ to get

$$\begin{aligned} & \sum_{P_0} \left[\left(\frac{\partial u_h}{\partial t}, v_h \right)_{K_{P_0}^*} - (u_h, L^* v_h)_{K_{P_0}^*} + (B u_h, v_h)_{\partial K_{P_0}^*} \right] \\ &= \sum_{P_0} (f, v_h)_{K_{P_0}^*}, \end{aligned} \quad (6.3.9)$$

where $u_h, v_h \in \bar{V}_h$, $B = \sum_{i=1}^2 \nu_i A_i$, and $\nu = (\nu_1, \nu_2)^T$ is the outer normal vector of $\partial K_{P_0}^*$. Assume that the m -order symmetric matrix B has m real eigenvalues: $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$, that there exists a constant $q \in [1, m]$ (dependent on $K_{P_0}^*$) and a constant $c_0 > 0$ (independent of $K_{P_0}^*$) such that

$$\lambda_1, \dots, \lambda_q < -c_0, \quad \lambda_{q+1}, \dots, \lambda_m > c_0, \quad (6.3.10)$$

and that there exists an m -order orthogonal matrix Q such that

$$B = Q \Lambda Q^T, \quad (Q^T = Q^{-1},) \quad (6.3.11)$$

where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$.

Set $w_h = (w_h^1, \dots, w_h^m)^T = Q^T u_h$ or $u_h = Q w_h$. For each side $\overline{Q_i Q_{i+1}}$ of $K_{P_0}^*$ (cf. Fig. 3.2.1), write q_i for the q validating (6.3.10), and define the upwind and the downwind values of w_h^j as

$$(w_h^j)^+ = \begin{cases} \text{the value of } w_h^j \text{ outside of } \overline{Q_i Q_{i+1}}, & \text{when } 1 \leq j \leq q_i, \\ \text{the value of } w_h^j \text{ inside of } \overline{Q_i Q_{i+1}}, & \text{when } q_i + 1 \leq j \leq m, \end{cases} \quad (6.3.12a)$$

$$(w_h^j)^- = \begin{cases} \text{the value of } w_h^j \text{ inside of } \overline{Q_i Q_{i+1}}, & \text{when } 1 \leq j \leq q_i, \\ \text{the value of } w_h^j \text{ outside of } \overline{Q_i Q_{i+1}}, & \text{when } q_i + 1 \leq j \leq m. \end{cases} \quad (6.3.12b)$$

Since

$$\begin{aligned} [w_h] &= ([w_h^1], \dots, [w_h^m]) \\ &= ((w_h^1)^+ - (w_h^1)^-, \dots, (w_h^m)^+ - (w_h^m)^-), \end{aligned}$$

we have

$$[u_h] = Q[w_h]. \quad (6.3.13)$$

Set $|\Lambda| = \text{diag}(|\lambda_1|, \dots, |\lambda_m|)$, and extend (6.2.9) to the system of equations here, then we have a semi-discrete highly accurate upwind difference scheme approximating (6.3.1): Find $u_h \in \bar{V}_h$ such that

$$\begin{aligned} \sum_{P_0} \left[\left(\frac{\partial u_h}{\partial t}, v_h \right)_{K_{P_0}^*} + (A \cdot \nabla u_h, v_h)_{K_{P_0}^*} + (K u_h, v_h)_{K_{P_0}^*} \right. \\ \left. - \frac{1}{2} (Q(|\Lambda| - \Lambda) Q^T [u_h], v_h)_{\partial K_{P_0}^*} \right] = \sum_{P_0} (f, v_h)_{K_{P_0}^*}, \end{aligned}$$

$$\forall v_h \in \bar{V}_h, \quad (6.3.14a)$$

$$(B - M)u_h = 0, \quad x \in \partial\Omega, \quad (6.3.14b)$$

$$u_h(x, 0) = \phi_h(x), \quad x \in \Omega. \quad (6.3.14c)$$

One can further approximate $\frac{\partial u_h}{\partial t}$ by a proper difference quotient to get forward, backward, or Crank-Nicolson fully-discrete highly accurate upwind schemes.

6.3.3 Estimation of a bilinear form

Let us introduce a bilinear form

$$\begin{aligned} a(u_h, v_h) &= \sum_{P_0} \left[(A \cdot \nabla u_h, v_h)_{K_{P_0}^*} + (K u_h, v_h)_{K_{P_0}^*} \right. \\ &\quad \left. - \frac{1}{2} (Q(|\Lambda| - \Lambda) Q^T [u_h], v_h)_{\partial K_{P_0}^*} \right]. \end{aligned} \quad (6.3.15)$$

By (6.3.7)

$$\begin{aligned} & (A \cdot \nabla v_h, v_h)_{K_{P_0}^*} + (K v_h, v_h)_{K_{P_0}^*} \\ &= \frac{1}{2} \left(\left(K + K^T - \sum_{i=1}^2 \frac{\partial A_i}{\partial x_i} \right) v_h, v_h \right)_{K_{P_0}^*} + \frac{1}{2} (B v_h, v_h)_{\partial K_{P_0}^*}. \end{aligned}$$

So we have

$$\begin{aligned} & a(v_h, v_h) \\ &= \frac{1}{2} \sum_{P_0} \left[\left(\left(K + K^T - \sum_{i=1}^2 \frac{\partial A_i}{\partial x_i} \right) v_h, v_h \right)_{K_{P_0}^*} + (Q \Lambda Q^T v_h, v_h)_{\partial K_{P_0}^*} \right. \\ & \quad \left. - (Q(|\Lambda| - \Lambda) Q^T [v_h], v_h)_{\partial K_{P_0}^*} \right]. \end{aligned} \quad (6.3.16)$$

It follows from the assumption (6.3.2b) that

$$\begin{aligned} a(v_h, v_h) &\geq \sigma_0 \sum_{P_0} (v_h, v_h)_{K_{P_0}^*} + \frac{1}{2} \sum_{P_0} [(Q^T Q \Lambda \tilde{v}_h, \tilde{v}_h)_{\partial K_{P_0}^*} \\ & \quad - (Q^T Q (|\Lambda| - \Lambda) [\tilde{v}_h], \tilde{v}_h)_{\partial K_{P_0}^*}] \\ &= \sigma_0 \sum_{P_0} (v_h, v_h)_{K_{P_0}^*} + \frac{1}{2} \sum_{P_0} [(\Lambda \tilde{v}_h, \tilde{v}_h)_{\partial K_{P_0}^*} \\ & \quad - ((|\Lambda| - \Lambda) [\tilde{v}_h], \tilde{v}_h)_{\partial K_{P_0}^*}], \end{aligned} \quad (6.3.17)$$

where $\tilde{v}_h = Q^T v_h$.

Let $\overline{Q_i Q_{i+1}}$ be a side of a dual element $K_{P_0}^*$ (cf. Fig. 3.2.1). On $\overline{Q_i Q_{i+1}}$, decompose $\tilde{v}_h$ into a sum of $\tilde{v}_h^-$ and $\tilde{v}_h^+$, where the first q_i entries of $\tilde{v}_h^-$ are equal to the counterpart of $\tilde{v}_h$ and the last $(m - q_i)$ entries are zero, while the last $(m - q_i)$ entries of $\tilde{v}_h^+$ are identical to those of $\tilde{v}_h$ and the first q_i entries are zero. So $\tilde{v}_h = \tilde{v}_h^- + \tilde{v}_h^+$. Also decompose Λ into a sum of Λ^- and Λ^+ , where $\Lambda^- = \text{diag}(\lambda_1, \dots, \lambda_{q_i}, 0, \dots, 0)$ and $\Lambda^+ = \text{diag}(0, \dots, 0, \lambda_{q_i+1}, \dots, \lambda_m)$.

Then, the second term on the right-hand side of (6.3.17) is equal to

$$\begin{aligned}
 J &= \sum_{P_0} \sum_i \int_{\overline{Q_i Q_{i+1}}} \left[\frac{1}{2} \langle \Lambda \tilde{v}_h, \tilde{v}_h \rangle - \frac{1}{2} \langle (|\Lambda| - \Lambda) [\tilde{v}_h], \tilde{v}_h \rangle \right] ds \\
 &= \sum_{P_0} \sum_i \int_{\overline{Q_i Q_{i+1}}} \left[\frac{1}{2} \langle \Lambda \tilde{v}_h, \tilde{v}_h \rangle - \frac{1}{2} \langle (|\Lambda| - \Lambda) [\tilde{v}_h], \tilde{v}_h \rangle \right. \\
 &\quad \left. + \frac{1}{4} \langle (|\Lambda| - \Lambda) [\tilde{v}_h], \tilde{v}_h \rangle - \frac{1}{4} \langle (|\Lambda| - \Lambda) [\tilde{v}_h], \tilde{v}_h \rangle \right] ds.
 \end{aligned} \tag{6.3.18}$$

But on $\overline{Q_i Q_{i+1}}$

$$\begin{aligned}
 \frac{1}{2} \langle \Lambda \tilde{v}_h, \tilde{v}_h \rangle &= \frac{1}{2} \langle \Lambda^- \tilde{v}_h^-, \tilde{v}_h^- \rangle + \frac{1}{2} \langle \Lambda^+ \tilde{v}_h^+, \tilde{v}_h^+ \rangle, \\
 -\frac{1}{2} \langle (|\Lambda| - \Lambda) [\tilde{v}_h], \tilde{v}_h \rangle &= \langle \Lambda^- (\tilde{v}_h^+ - \tilde{v}_h^-), \tilde{v}_h^- \rangle, \\
 -\frac{1}{4} \langle (|\Lambda| - \Lambda) [\tilde{v}_h], [\tilde{v}_h] \rangle &= \frac{1}{2} \langle \Lambda^- (\tilde{v}_h^+ - \tilde{v}_h^-), \tilde{v}_h^+ - \tilde{v}_h^- \rangle.
 \end{aligned}$$

The sum of the left-hand sides of the above three equalities are

$$\frac{1}{2} [\langle \Lambda^- \tilde{v}_h^+, \tilde{v}_h^+ \rangle + \langle \Lambda^+ \tilde{v}_h^+, \tilde{v}_h^+ \rangle].$$

Hence

$$\begin{aligned}
 J &= \frac{1}{2} \sum_{P_0} \sum_i \int_{\overline{Q_i Q_{i+1}}} [\langle \Lambda^- \tilde{v}_h^+, \tilde{v}_h^+ \rangle \\
 &\quad + \langle \Lambda^+ \tilde{v}_h^+, \tilde{v}_h^+ \rangle + \langle \Lambda^- \tilde{v}_h^-, \tilde{v}_h^- \rangle] ds.
 \end{aligned}$$

Notice that on the inner boundaries the first two integrals on the right-hand side of the above equality cancel each other out, and that on the boundary of Ω we have the zero boundary condition. Therefore we have

$$J = \frac{1}{2} \sum_{P_0} \sum_i \int_{\overline{Q_i Q_{i+1}}} \langle \Lambda^- \tilde{v}_h^-, \tilde{v}_h^- \rangle ds + \int_{\partial\Omega} \langle M \tilde{v}_h^-, \tilde{v}_h^- \rangle ds. \tag{6.3.19}$$

Write $\gamma_0 = \min(\sigma_0, \frac{1}{2})$ and

$$\|v_h\|_{0,\Omega}^2 = (v_h, v_h),$$

$$\|v_h\|_{0,\partial\Omega}^2 = \sum_{P_0} \sum_i \int_{Q_i Q_{i+1}} \langle |\Lambda^-| \tilde{v}_h^-, \tilde{v}_h^- \rangle ds + \int_{\partial\Omega} \langle M \tilde{v}_h^-, \tilde{v}_h^- \rangle ds.$$

Then it follows from (6.3.17)-(6.3.19) that

$$a(v_h, v_h) \geq \gamma_0 (\|v_h\|_{0,\Omega}^2 + \|v_h\|_{0,\partial\Omega}^2). \quad (6.3.20)$$

This shows the positive definiteness of the bilinear form $a(u_h, v_h)$.

Analogous to the proof in §6.2, one can obtain the error estimates like (6.2.30) for semi- and fully-discrete approximations.

6.3.4 Some practical difference schemes

Consider the first order linear equation:

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0, \quad 0 < x < L, \quad t > 0, \quad (6.3.21)$$

where a is a constant. Take a step size $h = L/N$ and nodes $x_j = jh$ ($0 \leq j \leq N$), then we have a uniform grid T_h :

$$0 = x_0 < x_1 < \cdots < x_N = L.$$

Then, choose dual nodes $x_{j+1/2} = (j + \frac{1}{2})h$ ($j = 0, 1, \dots, N-1$) to obtain a dual grid T_h^* :

$$0 = x_0 < x_{1/2} < x_{3/2} < \cdots < x_{N-1/2} < x_N = L.$$

The classical upwind scheme

Let the finite element space V_h be composed of piecewise constant functions relative to the dual grid. Obviously any $u_h \in V_h$ has the following expression:

$$u_h = u_h(x, t) = \sum_j u_j(t) \psi_j^{(0)}(x), \quad (6.3.22)$$

where

$$\psi_j^{(0)}(x) = \begin{cases} 1, & x \in [x_{j-1/2}, x_{j+1/2}], \\ 0, & \text{elsewhere,} \end{cases} \quad 1 \leq j \leq N-1,$$

$$\begin{aligned}\psi_0^{(0)}(x) &= \begin{cases} 1, & x \in [0, x_{1/2}], \\ 0, & \text{elsewhere,} \end{cases} \\ \psi_N^{(0)}(x) &= \begin{cases} 1, & x \in [x_{N-1/2}, x_N], \\ 0, & \text{elsewhere.} \end{cases}\end{aligned}\tag{6.3.23}$$

Evidently $u_h(x_j, t) = u_j(t)$. Substitute (6.3.22) in (6.2.9), and choose $v_h = \psi_j^{(0)}$, then we have

$$\begin{aligned}& \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial u_h}{\partial t} dx + \int_{x_{j-1/2}}^{x_{j+1/2}} a \frac{\partial u_h}{\partial x} dx \\& - \frac{1}{2} [(|a| - a)(u_{j+1}(t) - u_j(t)) \\& + (|a| + a)(u_{j-1}(t) - u_j(t))] \\& = h \frac{du_j(t)}{dt} - \frac{1}{2} [(|a| - a)(u_{j+1}(t) - u_j(t)) \\& + (|a| + a)(u_{j-1}(t) - u_j(t))] \\& = 0.\end{aligned}$$

Take a time step size $\tau > 0$ and exploit the forward difference formula

$$\frac{du_j(t)}{dt} \approx \frac{u_j^{n+1} - u_j^n}{\tau}, \quad u_j^n \approx u_j(n\tau),$$

then we have

$$\frac{u_j^{n+1} - u_j^n}{\tau} + a \frac{u_j^n - u_{j-1}^n}{\tau} = 0, \quad \text{as } a \geq 0, \tag{6.3.24a}$$

$$\frac{u_j^{n+1} - u_j^n}{\tau} + a \frac{u_{j+1}^n - u_j^n}{\tau} = 0, \quad \text{as } a < 0. \tag{6.3.24b}$$

This is precisely the classical upwind scheme. The sufficient and necessary condition of its stability is $r = |a|\tau/h \leq 1$ (cf. [A-27] and [B-74]).

A second order upwind scheme

Let the basis functions of V_h be composed of two groups of functions, of which the first group is $\{\psi_j^{(0)}\}$ given in (6.3.23), and the other one is $\{\psi_j^{(1)}\}$ defined by

$$\psi_j^{(1)}(x) = \begin{cases} x - x_j, & x \in [x_{j-1/2}, x_{j+1/2}], \\ 0, & \text{elsewhere,} \end{cases} \quad 1 \leq j \leq N-1,$$

$$\psi_0^{(1)}(x) = \begin{cases} x, & x \in [0, x_{1/2}], \\ 0, & \text{elsewhere,} \end{cases}$$

$$\psi_N^{(0)}(x) = \begin{cases} x - x_N, & x \in [x_{N-1/2}, x_N], \\ 0, & \text{elsewhere.} \end{cases}$$

An element of V_h is of the form

$$u_h = u_h(x, t) = \sum_j [u_{0j}(t)\psi_j^{(0)}(x) + u_{1j}(t)\psi_j^{(1)}(x)]. \quad (6.3.26)$$

Substitute u_h into (6.2.9), choose $v_h = \psi_j^{(0)}$ and $\psi_j^{(1)}$, and approximate $\frac{\partial u_h}{\partial t}$ by a forward differencing, then we have the following two groups of equations:

$$\begin{cases} \frac{u_{0j}^{n+1} - u_{0j}^n}{\tau} + a \frac{u_{0j}^n - u_{0j-1}^n}{h} + \frac{a}{2}(u_{1j}^n - u_{1j-1}^n) = 0, & \text{as } a \geq 0, \\ \frac{u_{0j}^{n+1} - u_{0j}^n}{\tau} + a \frac{u_{0j+1}^n - u_{0j}^n}{h} + \frac{a}{2}(u_{1j+1}^n - u_{1j}^n) = 0, & \text{as } a < 0, \end{cases} \quad (6.3.27a)$$

$$\begin{cases} \frac{u_{1j}^{n+1} - u_{1j}^n}{\tau} - \frac{6a}{h^2}(u_{0j}^n - u_{0j-1}^n) + \frac{3a}{h}(u_{1j}^n + u_{1j-1}^n) = 0, & \text{as } a \geq 0, \\ \frac{u_{1j}^{n+1} - u_{1j}^n}{\tau} - \frac{6a}{h^2}(u_{0j+1}^n - u_{0j}^n) + \frac{3a}{h}(u_{1j+1}^n + u_{1j}^n) = 0, & \text{as } a < 0. \end{cases} \quad (6.3.27b)$$

One may use the variable separation method to deduce the corresponding amplification matrix:

$$G = \begin{bmatrix} 1 - \frac{a\tau}{h}(1 - e^{-i\sigma h}) & -\frac{a\tau}{2h}(1 - e^{-i\sigma h}) \\ \frac{6a\tau}{h^2}(1 - e^{-i\sigma h}) & 1 - \frac{3a\tau}{h}(1 + e^{-i\sigma h}) \end{bmatrix},$$

where $\sigma = 2\pi l$, ($l = 0, \pm 1, \dots$), $i = \sqrt{-1}$. It can be proved that G does not satisfy the von Neumann condition, since it always has an eigenvalue such that the lower bound of the absolute value of this eigenvalue is greater than 1, so (6.3.27) is absolutely unstable. But if we instead use backward or Crank-Nicolson difference approximations, then the resulting upwind schemes will be absolutely stable.

6.3.5 A numerical example

We use upwind schemes to solve the Riemann problem of Burger's equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x}(u^2) = 0, \quad -\infty < x < \infty, \quad (6.3.28a)$$

$$u(x, 0) = \begin{cases} 1, & \text{as } x \leq 0, \\ 0, & \text{as } x > 0. \end{cases} \quad (6.3.28b)$$

The classical upwind scheme leads to the equations:

$$\begin{cases} u_j^{n+1} = u_j^n - ru_j^n(u_j^n - u_{j-1}^n), & \text{as } u_j^n \geq 0, \\ u_j^{n+1} = u_j^n - ru_j^n(u_{j+1}^n - u_j^n), & \text{as } u_j^n < 0. \end{cases} \quad (6.3.29)$$

where $r = \tau/h$. The numerical solution is given in Fig. 6.3.1(a). We observe that the shock wave is too flat and evolves too slow, indicating a notable error. Fig. 6.3.1(b) depicts the numerical results of the second order Crank-Nicolson upwind scheme (cf. [B-60]). Now the shock wave is steeper and its evolution is faster, very close to the true solution. But there appear oscillations after the wave.

Remark If we keep working with only one grid T_h , and replace the dual element by an element $K \in T_h$, then our strategy of

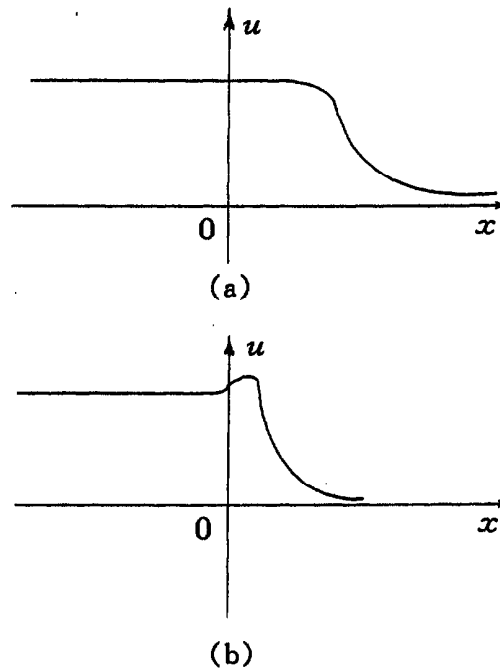


Fig. 6.3.1

constructing the generalized upwind scheme still works and results in some difference schemes called *box schemes*. Parallel results of convergence and error estimates can be similarly obtained for these box schemes (cf. [A-59] and [B-41]).

6.4 Finite Volume Methods for Nonlinear Conservative Hyperbolic Equations

The finite volume method (FVM for short), combined with, e.g., the Godunov scheme (see Example 1 below) or the TVD scheme (cf. [B-36]), has become one of the most popular methods for fluid

computation in the last twenty years. In this section we introduce a FVM for the following system of conservative hyperbolic equations:

$$\frac{\partial u_j}{\partial t} + \sum_{i=1}^n \frac{\partial f_{ij}}{\partial x_i} = 0, \quad j = 1, \dots, d, \quad (6.4.1)$$

where f_{ij} 's are smooth functions of $\mathbf{u} = (u_1, \dots, u_d)$. If we set $F = (f_{ij})_{n \times d}$ and $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$, then (6.4.1) becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot F = 0. \quad (6.4.1)'$$

An example of (6.4.1) is the Euler equation describing a one-dimensional non-steady-state flow ($n = 1, d = 3$):

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho u \\ \rho(e + \frac{1}{2}u^2) \end{pmatrix}, \quad F = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ \rho u(e + \frac{p}{\rho} + \frac{1}{2}u^2) \end{pmatrix},$$

where ρ is the density, p the pressure, u the velocity, and e the internal energy. ρ , p and e satisfy a state equation $p = p(\rho, e)$. Let $\Omega \subset R^n$ be a bounded region. The initial-boundary value problem of the conservative equation (6.4.1) reads: Find $\mathbf{u}(x, t) : \Omega \times [0, \infty) \rightarrow R^d$ satisfying (6.4.1) and

$$\begin{cases} \mathbf{u}(x, 0) = \mathbf{u}_0(x), & x \in \Omega, \\ \mathbf{u}(x, t) = \phi(x), & x \in \partial\Omega. \end{cases} \quad (6.4.2)$$

Next, we consider the case of $d = 1$ and $n = 2$ to illustrate the idea. Now (6.4.1) reads

$$\frac{\partial u}{\partial t} + \nabla \cdot F(u) = 0, \quad \text{on } \Omega \subset R^2, \quad (6.4.3)$$

where $F = (f_1, f_2)^T$. Let D be a subregion, e.g., a polygon, of Ω . Integrate (6.4.3) on D and make use of Green's formula, then we have

$$\int_D \frac{\partial u}{\partial t} dx + \int_{\partial D} F \cdot \nu ds = 0, \quad (6.4.3)'$$

where ν is the unit outer normal vector. This is an integral form of (6.4.3). The so-called finite volume method is precisely the discretization method based on (6.4.3)'.

Since (6.4.3) is a conservative equation, it is natural to expect, and we shall try to ensure, its discretization to possess the conservative property as well.

As before, we place a quasi-uniform triangulation $T_h = \{K\}$ on Ω . Suppose none of the triangular elements is an obtuse triangle. Denote by T_h^* the circumcenter dual grid of T_h . As in Fig. 3.2.2, let P_0 be a node of T_h , P_i ($i = 1, 2, \dots, 6$) the neighbouring nodes of P_0 , and $K_{P_0}^*$ the dual element surrounding P_0 with vertexes Q_i ($i = 1, 2, \dots, 6$). Choose in (6.4.3)' $D = K_{P_0}^*$ as a control volume, and set $t = n\tau$ ($\tau > 0$ is the time step size), then we have

$$\int_{K_{P_0}^*} \frac{\partial u^n}{\partial t} dx + \sum_{i=1}^6 \int_{\overline{Q_i Q_{i+1}}} F^n \cdot \nu ds = 0, \quad (Q_7 = Q_1) \quad (6.4.4)$$

where the superscript n denotes the function value on $t = t_n$. Note that $\overline{Q_i Q_{i+1}}$ is the perpendicular bisector of $\overline{P_0 P_{i+1}}$. So if we use $\nu_{0,i+1}$ to denote the unit outer normal vector on $\overline{Q_i Q_{i+1}}$ towards P_{i+1} , then

$$\mathcal{F}_{0,i+1} = \int_{\overline{Q_i Q_{i+1}}} F^n \cdot \nu_{0,i+1} ds \quad (6.4.5)$$

is the flux flowing out of $K_{P_0}^*$ and passing through the side $\overline{Q_i Q_{i+1}}$. Similarly consider the dual element $K_{P_{i+1}}^*$ surrounding P_{i+1} , then we have the flux out of $K_{P_{i+1}}^*$ and passing through $\overline{Q_i Q_{i+1}}$:

$$\mathcal{F}_{i+1,0} = \int_{\overline{Q_i Q_{i+1}}} F^n \cdot \nu_{i+1,0} ds.$$

Apparently

$$\mathcal{F}_{i+1,0} = -\mathcal{F}_{0,i+1}.$$

Now we can write (6.4.4) in the form

$$\int_{K_{P_0}^*} \frac{\partial u^n}{\partial t} dx + \sum_{i=1}^6 \mathcal{F}_{0,i+1} = 0. \quad (6.4.6)$$

The time derivative is usually approximated by a forward (explicit) differencing. In the space direction, one often uses u_0^n and u_{i+1}^n to discretize (6.4.5) to obtain a so-called numerical flux. So let the numerical flux be of the form

$$\mathcal{F}_{0,i+1}^h = |\overline{Q_i Q_{i+1}}| g_{0,i+1}(u_0^n, u_{i+1}^n), \quad (6.4.7)$$

where $g_{0,i+1}(u_0^n, u_{i+1}^n)$ is suitably chosen. Then we have the finite volume equation:

$$u_0^{n+1} = u_0^n - \sum_i \frac{\tau |\overline{Q_i Q_{i+1}}|}{S_{P_0}^*} g_{0,i+1}(u_0^n, u_{i+1}^n), \quad (6.4.8)$$

where $S_{P_0}^*$ denotes the area of the dual element $K_{P_0}^*$. We require $g_{0,i+1}(u_0^n, u_{i+1}^n)$ to satisfy the conservative property

$$g_{0,i+1}(u_0^n, u_{i+1}^n) = -g_{i+1,0}(u_{i+1}^n, u_0^n) \quad (6.4.9)$$

and the consistency (cf. (6.4.3)')

$$g_{0,i+1}(u, u) = F(u) \cdot \nu_{0,i+1}. \quad (6.4.10)$$

We also require $g_{0,i+1}(u_0^n, u_{i+1}^n)$ to possess a monotonicity, that is,

$$\frac{\partial g_{0,i+1}}{\partial u_0} \geq 0, \quad \frac{\partial g_{0,i+1}}{\partial u_{i+1}} \leq 0. \quad (6.4.11)$$

The numerical flux $g_{0,i+1}$ can be determined by, e.g., one-dimensional Godunov or Lax-Friedrichs schemes. In fact, if we define an x' -axis along $\overline{P_0 P_{i+1}}$ with positive direction $\nu_{0,i+1}$, let any a point of $x \in R^2$ on $\overline{P_0 P_{i+1}}$ correspond to $x' = x \cdot \nu_{0,i+1} \in R$, and set $w = w(x', t) = u(x, t)$, then $g_{0,i+1}$ becomes the numerical flux consistent with the following one-dimensional equation:

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x'} (F(w) \cdot \nu_{0,i+1}) = 0. \quad (6.4.12)$$

Example 1. Godunov scheme. Set $f_{0,i+1} = F(w) \cdot \nu_{0,i+1}$, and define

$$g_{0,i+1} = f_{0,i+1}(w(0^+; u_0, u_{i+1}; f_{0,i+1})),$$

where $w(0^+; u_0, u_{i+1}; f_{0,i+1}) = w(x', t)$ is the solution of the following Riemann problem

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x'} f_{0,i+1}(w) = 0, \quad t > 0, \quad x' \in R,$$

$$w(x', 0) = \begin{cases} u_0, & \text{when } x' < 0, \\ u_{i+1}, & \text{when } x' > 0. \end{cases}$$

Here we require the time step size to be small enough such that the waves of the neighbouring Riemann problems will not interfere each other. One may show that the Godunov scheme indeed possesses properties (6.4.9)-(6.4.11). (cf. [B-19].)

Example 2. Lax-Friedrichs scheme. In this case the numerical flux is defined by

$$g_{0,i+1} = \frac{1}{2}(\nu_{0,i+1}F(u_0) + \nu_{0,i+1}F(u_{i+1})) - \frac{1}{2\lambda_{0,i+1}}(u_{i+1} - u_0),$$

where $\lambda_{0,i+1}$ is independent of u_0 and u_{i+1} , and satisfies

$$\lambda_{0,i+1} = \lambda_{i+1,0} > 0$$

as well as the CFL condition

$$\lambda_{0,i+1} \left\| \frac{\partial}{\partial u} F \cdot \nu_{0,i+1} \right\|_{\infty} \leq 1.$$

It can be shown (see [B-19] and [B-12]) that this scheme has the properties (6.4.9)-(6.4.11).

Example 3 Suppose F in equation (6.4.3) is of the form

$$F = b(x, t)f(u),$$

where $f : R \rightarrow R$ is a smooth function and $b(x, t) : R^2 \times [0, \infty) \rightarrow R^2$ a given vector function. Then (6.4.4) reads

$$\int_{K_{P_0}^*} \frac{\partial u^n}{\partial t} dx + \sum_{i=1}^6 \int_{\bar{Q}_i \bar{Q}_{i+1}} (b \cdot \nu) f(u^n) ds = 0.$$

Set

$$\begin{aligned}\beta_{ij} &= \frac{1}{|Q_i Q_{i+1}|} \int_{Q_i Q_{i+1}} b \cdot \nu ds, \\ g_{0,i+1} &= \begin{cases} \beta_{ij} f(u_{i+1}), & \text{when } \beta_{ij} \leq 0, \\ \beta_{ij} f(u_0), & \text{when } \beta_{ij} > 0, \end{cases} \\ \mathcal{F}_{0,i+1}^h &= |Q_i Q_{i+1}| g_{0,i+1}(u_0^n, u_{i+1}^n).\end{aligned}$$

Then (6.4.8) is precisely the upwind scheme given in §6.2. Clearly when τ is sufficiently small such that the CFL condition holds, the upwind scheme satisfies the conditions (6.4.9) and (6.4.11). The consistency condition is now valid approximately.

Remark 1 The FVM considered here adopts the nodes of T_h as the center of the control volume, and hence is referred to be node-centred. Actually a node of T_h is also a center of an element of T_h^* . So this FVM can as well be called cell-centred with respect to the dual grid T_h^* . Another strategy is to take a cell $K \in T_h$ (or $K^* \in T_h^*$) as the control center, and the vertexes of K (or K^*) as nodes, resulting a FVM of cell-vertex type. (cf. [B-79].)

Remark 2 FVM can be extended to solve a system of equations ($d > 1$). For instance, let us consider the case $n = 1$. (6.4.1) now reads

$$\frac{\partial \mathbf{u}}{\partial t} + F'(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x} = 0.$$

Since we assume that this system of equations is of hyperbolic type, the Jacobian matrix $A = F'(\mathbf{u})$ can be reduced to a diagonal form. Accordingly, we can deduce this system to a system of scalar characteristic equations:

$$\frac{\partial w^k}{\partial t} + a^k \frac{\partial w^k}{\partial x} = 0, \quad k = 1, 2, \dots, d.$$

Now we can employ the FVM to each equation to obtain a numerical flux on x -direction ([B-36]). Furthermore, in the two-dimensional case ($n = 2$), one can similarly get the numerical flux on y -direction, and to deduce a FVM on a rectangular grid.

Remark 3. A scheme satisfying (6.4.11) is called a monotone scheme. There has been a great deal of research on the convergence of monotone FVM (cf. [A-57], [B-40], [B-19] and [B-12] etc.).

Remark 4. In [B-97], FVM has been successfully used to generalize TVD schemes to constructive quadrilateral grids, which have found wide applications in aerodynamics computations. (cf. [B-43,44,75,80,92].)

Bibliography and Comments

[A-1,24] are among the earliest works for second order hyperbolic equations. An abstract framework is constructed in [A-1]. [A-24] deals with quasi-linear hyperbolic equations. The first section of this chapter is based upon [A-24]. An extension of the results in [A-24] to more general quasi-linear hyperbolic equations is presented in [A-45,47].

The second and third sections of this chapter are devoted to the study of generalized difference methods for first order system of hyperbolic equations, according in principle to [B-60]. Based on discontinuous finite element methods ([B-53],[A-59]), high accuracy generalized difference methods are proposed in [B-60], which differ from the usual discontinuous finite element methods in the following two aspects. Firstly, the discontinuous finite element methods are now used on the dual grid T_h^* rather than the original grid T_h . Only in this way, one may end up with an extension of the upwind scheme. Secondly, an artificial viscosity, instead of the least square method as usual, is used in the extension of the results from a equation to a system of equations [B-41].

The high accuracy methods require further improvements such as: How to reduce the superfluous oscillations? And how to modify them for computing the discontinuous solutions (shock waves) of quasi-linear conservative equations. It seems necessary to introduce a proper diffusion term in the schemes.

The finite volume method was first used for computational fluid dynamics in the early seventies, resulting in a great number of references as well as software applications. §6.4 is only a rough introduc-

tion to this topic. For details, please see the corresponding references at the end of the book and certain journals such as J. Comput. Phys. and AIAA Journal.

Chapter 7

CONVECTION- DOMINATED DIFFUSION PROBLEMS

Convection-dominated diffusion problems often arise in mechanics, physics and other disciplines of applications. They are parabolic (non-steady-state) or elliptic (steady-state) equations. There have been many papers in recent years devoted to the numerical solution of this class of equations, aiming at constructing schemes that are stable, highly accurate, and suitable for small diffusion coefficients. Up to now, the schemes have been mainly various kinds of combinations of finite difference or finite element methods and characteristic methods (cf. [B-26]). In this chapter, we introduce some combinations of difference or generalized difference methods and characteristic methods, i.e., we use difference or generalized difference methods to discretize the diffusion term and characteristic methods to the convection term, resulting in various kinds of extensions of upwind schemes.

7.1 One-Dimensional Characteristic Difference Schemes

Consider the following Cauchy problem of a one-dimensional non-steady state convection-diffusion equation:

$$\begin{cases} \frac{\partial u}{\partial t} + b(x) \frac{\partial u}{\partial x} - \mu \frac{\partial^2 u}{\partial x^2} = f(x), & x \in \mathbb{R}, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}. \end{cases} \quad (7.1.1a) \quad (7.1.1b)$$

We assume $\mu > 0$ is a constant, and $\min_x |b(x)|$ is very large in comparison with μ . Write

$$\psi(x) = [1 + b^2(x)]^{1/2}. \quad (7.1.2)$$

The characteristic direction with respect to the operator $\frac{\partial u}{\partial t} + b(x) \frac{\partial u}{\partial x}$ is

$$\tau = \tau(x) = \left(\frac{1}{\psi(x)}, \frac{b(x)}{\psi(x)} \right).$$

The directional derivative along τ is

$$\frac{\partial}{\partial \tau(x)} = \frac{1}{\psi(x)} \frac{\partial}{\partial t} + \frac{b(x)}{\psi(x)} \frac{\partial}{\partial x}. \quad (7.1.3)$$

Thus (7.1.1a) can be written as

$$\psi(x) \frac{\partial u}{\partial \tau} - \mu \frac{\partial^2 u}{\partial x^2} = f(x), \quad x \in \mathbb{R}, t > 0. \quad (7.1.4)$$

Take a time step size $\Delta t > 0$, and place a grid on t -axis with nodes $t_n = n\Delta t$ ($n = 0, 1, \dots$). The characteristic direction starting from (x, t_n) crosses the straight line $t = t_{n-1}$ at

$$\bar{x} = x - b(x)\Delta t. \quad (7.1.5)$$

Naturally we use the following formula to approximate the characteristic directional derivative:

$$\begin{aligned} \psi(x) \frac{\partial u}{\partial \tau} &\approx \psi(x) \frac{u(x, t_n) - u(\bar{x}, t_{n-1})}{[(x - \bar{x})^2 + (\Delta t)^2]^{1/2}} \\ &= \frac{u(x, t_n) - u(\bar{x}, t_{n-1})}{\Delta t}. \end{aligned} \quad (7.1.6)$$

Correspondingly (7.1.4) is approximated by

$$\frac{u^n(x) - u^{n-1}(\bar{x})}{\Delta t} - \mu \frac{\partial^2 u^n(x)}{\partial x^2} = f(x). \quad (7.1.7)$$

It remains to discretize the space variable. Since $\bar{x}$ is not necessarily a node, we need to evaluate the approximate solution $u_h(\bar{x}, t_{n-1})$. This is an easy matter for Galerkin finite element methods or generalized difference methods. As for finite difference methods, a linear or quadratic interpolation in terms of nodal values is often adopted to compute $u_h(\bar{x}, t_{n-1})$.

7.1.1 Difference methods based on algebraic interpolations

Take a space step size $h > 0$ and nodes $x_i = ih$. Noticing (7.1.5), we set $\bar{x}_i = x_i - b_i \Delta t$, where $b_i = b(x_i)$. Denote by u_i^n a nodal function and $u^n(x)$ the piecewise linear function with nodal values u_i^n . Let $\bar{u}_i^n = u^n(\bar{x}_i)$. Define the second order central difference quotient

$$\bar{\partial}_{xx} u_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}.$$

Then, the simplest difference scheme approximating (7.1.1) or (7.1.7) is:

$$\begin{cases} \frac{u_i^n - \bar{u}_{i-1}^n}{\Delta t} - \mu \bar{\partial}_{xx} u_i^n = f_i^n, & i = 0, \pm 1, \dots; n \geq 1, \\ u_i^0 = u_0(x_i), & i = 0, \pm 1, \dots \end{cases} \quad (7.1.8a)$$

$$(7.1.8b)$$

This is an implicit characteristic difference scheme. One can use it to compute u_i^n iteratively, starting from the initial value u_i^0 .

Let $u(x, t)$ be the solution to (7.1.1). Restricting it at the nodes and inserting it into (7.1.8a) yield

$$\frac{u(x_i, t_n) - u(\bar{x}_i, t_{n-1})}{\Delta t} - \mu \bar{\partial}_{xx} u(x_i, t_n) = f_i^n + r_i^n, \quad (7.1.9)$$

where r_i^n is the truncation error. A simple calculation gives

$$r_i^n = \frac{(1 + b_i^2)}{2} \frac{\partial^2 u^*}{\partial \tau^2} \Delta t + O(\|u(x, t_{n-1})\|_{3, \infty} h), \quad (7.1.10)$$

$$(\text{or } O(\|u(x, t_{n-1})\|_{4,\infty} h^2))$$

where $\frac{\partial^2 u^*}{\partial \tau^2}$ is the tangent-directional second derivative of u along the characteristic line segment between (x_i, t_n) and $(\bar{x}_i, t_{n-1})$. When $\mu = 0$, u varies linearly as τ . If in addition we assume $f = 0$, then u becomes a constant along the characteristic line. Hence in the convection-dominated case, the second derivative $\frac{\partial^2 u}{\partial \tau^2}$ is generally less than $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$. Suppose $b(x)$ is bounded: $|b(x)| \leq K$, then

$$|r_i^n| \leq \frac{1}{2}(1 + K^2)\Delta t \sup_x \left| \frac{\partial^2 u^*}{\partial \tau^2} \right| + Ch\|u(x, t_n)\|_{3,\infty}, \quad (7.1.11)$$

where C is the general constant. Therefore, the order of the truncation error is $O(\Delta t + h)$.

If we wish to obtain an error order $O(\Delta t + h^2)$, then we have to use a quadratic interpolation. In such a case we naturally require

$$\Delta t = O(h^2), \text{ as } h \rightarrow 0.$$

By virtue of $|b(x)| \leq K$ and (7.1.5), we know that, for sufficiently small Δt , $\bar{x}_i$ will lie in between x_{i-1} and x_{i+1} . So we may use u_{i-1}^{n-1} , u_i^{n-1} and u_{i+1}^{n-1} to get a quadratic interpolation function $u^{n-1}(x)$ so as to determine $u^{n-1}(\bar{x}_i)$:

$$u^{n-1}(\bar{x}_i) = \frac{1}{2}\alpha_i^2(u_{i+1}^{n-1} + u_{i-1}^{n-1}) + (1 - \alpha_i^2)u_i^{n-1} + \frac{1}{2}\alpha_i(u_{i+1}^{n-1} - u_{i-1}^{n-1}), \quad (7.1.12)$$

where

$$\alpha_i = -b_i \Delta t / h. \quad (7.1.13)$$

Then we can similarly obtain a difference equation as (7.1.8), save that $\bar{u}_i^{n-1} = u^{n-1}(\bar{x}_i)$ is now computed according to (7.1.12). A simple calculation shows that the truncation error in (7.1.9) now reads:

$$|r_i^n| \leq C \left(\Delta t \sup_{x,n} \left| \frac{\partial^2 u^*}{\partial \tau^2} \right| + h^2 \sup_{t>0} \|u(x, t)\|_{4,\infty} \right). \quad (7.1.14)$$

Its order is $O(\Delta t + h^2)$.

7.1.2 Upwind difference schemes

Once again consider the difference scheme (7.1.8a), where $\bar{u}_i^{n-1} = u^{n-1}(\bar{x}_i)$, $u^{n-1}(x)$ is the piecewise linear function with nodal values u_i^{n-1} , and $\bar{x}_i = x_i - b_i \Delta t$. If $b_i \geq 0$ and Δt is sufficiently small, then $\bar{x}_i \in [x_{i-1}, x_i]$ and in such a case

$$\bar{u}_i^{n-1} = \frac{b_i \Delta t}{h} u_{i-1}^{n-1} + \frac{h - b_i \Delta t}{h} u_i^{n-1}.$$

So (7.1.8a) can be written as

$$\frac{u_i^n - u_i^{n-1}}{\Delta t} + b_i \frac{u_i^{n-1} - u_{i-1}^{n-1}}{h} - \mu \bar{\delta}_{xx} u_i^n = f_i^n. \quad (7.1.15)$$

This amounts to approximating the convection term by a backward differencing. Similarly, a forward differencing should be adopted to approximate the convection term when $b_i < 0$. Therefore, the difference equation (7.1.8) based on linear interpolations is a kind of upwind scheme. We recall that the usual upwind scheme takes simultaneously the differencing of convection and diffusion terms on either $(n-1)$ or n level, resulting in explicit or implicit schemes respectively.

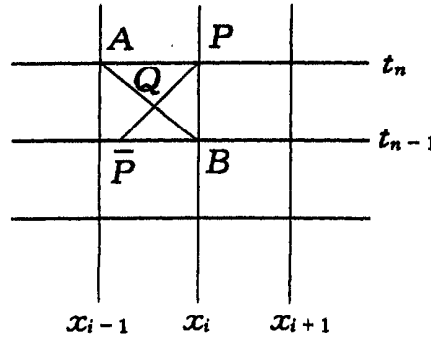


Fig. 7.1.1

Now we try to replace the explicit difference approximation (7.1.6) by an implicit one. Let a grid be given as in Fig. 7.1.1, where the nodes $P = (x_i, t_n)$, $A = (x_{i-1}, t_n)$ and $B = (x_i, t_{n-1})$. If $b_i \geq 0$, then the characteristic line starting from P crosses the net line $t = t_{n-1}$ at

$\bar{P} = (\bar{x}_i, t_{n-1})$ on the left-hand side of B . Let the cross point of $\overline{P\bar{P}}$ and the diagonal line $\overline{AB}$ be $Q = (x', t')$. Take u_A and u_B as nodal values to construct a linear interpolation along $\overline{AB}$, and notice

$$\frac{|\overline{AQ}|}{|\overline{QB}|} = \frac{|\overline{AP}|}{|\overline{PB}|} = \frac{h}{b_i \Delta t}.$$

Then we have

$$u_Q = \frac{b_i \Delta t}{h + b_i \Delta t} u_A + \frac{h}{h + b_i \Delta t} u_B = \frac{b_i \Delta t}{h + b_i \Delta t} u_{i-1}^n + \frac{h}{h + b_i \Delta t} u_i^{n-1}. \quad (7.1.16)$$

Note $|\overline{P\bar{P}}| = \sqrt{\Delta t^2 + b_i^2 \Delta t^2} = \psi \Delta t$ (cf. (7.1.2)). So it follows from the similarity of $\triangle APQ$ and $\triangle BQ\bar{P}$ that

$$\frac{b_i \Delta t}{h} = \frac{|\overline{PQ}|}{|\overline{PQ}|} = \frac{|\overline{P\bar{P}}|}{|\overline{PQ}|} - 1 = \frac{\psi \Delta t}{|\overline{PQ}|} - 1.$$

Thus

$$|\overline{PQ}| = \psi h \Delta t / (h + b_i \Delta t). \quad (7.1.17)$$

Let us employ the following approximation instead of (7.1.6)

$$\psi(x) \frac{\partial u}{\partial \tau} \approx \psi(x) \frac{u_P - u_Q}{|\overline{PQ}|}.$$

Substituting (7.1.16) and (7.1.17) in the right-hand side yields

$$\psi(x_i) \frac{\partial u}{\partial \tau} \approx \frac{u_i^n - u_i^{n-1}}{\Delta t} + b_i \frac{u_i^n - u_{i-1}^n}{h}.$$

Finally we end up with an implicit upwind scheme for $b_i \geq 0$:

$$\frac{u_i^n - u_i^{n-1}}{\Delta t} + b_i \frac{u_i^n - u_{i-1}^n}{h} - \mu \bar{\partial}_{xx} u_i^n = f_i^n. \quad (7.1.18)$$

When $b_i < 0$, the convection term on the left-hand side should be approximated by a forward differencing.

For a steady-state problem

$$b(x) \frac{\partial u}{\partial x} - \mu \frac{\partial^2 u}{\partial x^2} = f(x), \quad (7.1.19)$$

a backward differencing should be adopted to approximate the convection term when $b_i \geq 0$, and a forward differencing when $b_i < 0$. Now let us consider the extension to multidimensional cases. Take the following two-dimensional convection-diffusion equation as an example:

$$\frac{\partial u}{\partial t} + b_1 \frac{\partial u}{\partial x_1} + b_2 \frac{\partial u}{\partial x_2} - \mu \Delta u = f, \quad (7.1.20)$$

where $\mu > 0$ and $\|b\| = \|(b_1, b_2)\| \gg \mu$. The character is now a curve

$$\frac{dx_1}{dt} = b_1(x), \quad \frac{dx_2}{dt} = b_2(x).$$

Place a (triangular or rectangular) grid on x -plane, and a uniform grid on t -axis with a time step size $\Delta t > 0$. Assume that the characteristic line at a node (x_{1i}, x_{2j}, t_n) crosses the plane $t = t_{n-1}$ at point $\bar{x}_{ij} = (\bar{x}_{1i}, \bar{x}_{2j})$. Evaluate $\bar{u}_{ij}^{n-1}$ as an interpolation in terms of certain neighbouring nodes of $\bar{x}_{ij}$. Then one may perform further discretization analogously as in the one-dimensional case. In particular, if a rectangular grid is used on x -plane, then the value of $\bar{u}_{ij}^{n-1}$ can be obtained by either a piecewise bilinear, or a biquadratic interpolation. One may also approximate $b_1 \partial u / \partial x_1$ (resp. $b_2 \partial u / \partial x_2$) along x_1 -axis (resp. x_2 -axis) as a one-dimensional convection term. Another strategy is to use the alternating direction method or the locally one-dimensional scheme to reduce the two-dimensional problem into one-dimensional problems along different directions, and then to further discretize the resulting one-dimensional convection terms.

7.2 Generalized Upwind Difference Schemes for Steady-state Problems

Let us consider a steady-state convection-diffusion problem:

$$\begin{cases} -\mu \Delta u + b \cdot \nabla u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (7.2.1a) \quad (7.2.1b)$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal region, $\Gamma = \partial\Omega$ is the boundary of Ω , $x = (x_1, x_2)$, $\mu > 0$ is the diffusion coefficient, and $b = b(x) = (b_1(x), b_2(x))$ is the convection velocity. By convection-dominated we

mean $0 < \mu \ll \|b\|_\infty$. In this section we extend, from another angle, upwind schemes to generalized upwind difference schemes.

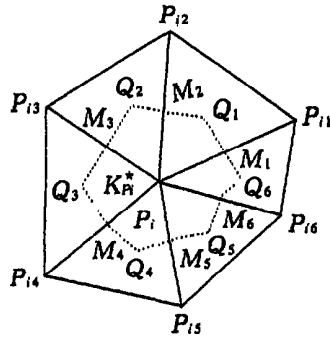


Fig. 7.2.1

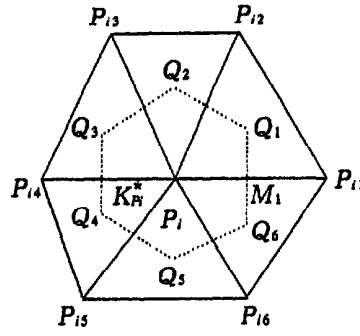


Fig. 7.2.2

7.2.1 Construction of the difference schemes

Place a suitable triangulation $T_h = \{K\}$ on Ω , $K \in T_h$ being a triangular element. Let $\{P_i\}_{i=1}^M$ be the set of grid nodes, where $\{P_i\}_{i=1}^N$ are inner nodes and $\{P_i\}_{N+1}^M$ are boundary nodes. Use $h(K)$ and $\rho(K)$ respectively to denote the maximum side length and the diameter of the inscribed circle, and set $h = \max_{K \in T_h} h(K)$. As usual we assume that T_h is a quasi-uniform grid, namely there exist constants $\gamma_1, \gamma_2 > 0$ such that

$$h(K)/\rho(K) \leq \gamma_1, \quad h(K)/h \geq \gamma_2, \quad \forall K \in T_h. \quad (7.2.2)$$

Obviously we have $\Omega_h \equiv \bigcup_{K \in T_h} K = \Omega$.

As in Chapter 3, we construct a barycenter or circumcenter dual grid T_h^* of T_h . For any node P_i (cf. Figg. 7.2.1 and 7.2.2), let P_{ij} ($1 \leq j \leq 6$) be the neighbouring nodes of P_i , M_j the midpoint of $\overline{P_i P_{ij}}$, Q_i the barycenter (Fig. 7.2.1) or circumcenter (cf. Fig. 7.2.2 where none of the elements is an obtuse triangle) of $\triangle P_i P_{ij} P_{ij+1}$. Connect successively $M_1, Q_1, M_2, Q_2, \dots, M_6, Q_6$ and M_1 to form a polygon $K_{P_i}^*$ surrounding P_i , called a dual element. The entire dual

elements constitute a new grid $T_h^* = \{K_{P_i}^*, 1 \leq i \leq M\}$ on Ω , referred to as a barycenter or a circumcenter dual grid.

Let us introduce the following trial function space:

$$U_h = \{u_h(x) \in C(\Omega) : u_h(x)|_K \text{ is linear}, u_h(x)|_\Gamma = 0\}.$$

A basis function $\phi_i(x)$ of U_h is equal to 1 at the inner node P_i and 0 at other nodes. So any $u_h \in U_h$ has the following expression:

$$u_h(x) = \sum_{i=1}^N u_h(P_i) \phi_i(x).$$

The test function space V_h is chosen as the piecewise constant function space related to T_h^* , subject to the boundary condition that $v_h(x) = 0$ on $K_{P_i}^*$ for any boundary node P_i and any $v_h \in V_h$. Let $\psi_i(x)$ be the characteristic function of $K_{P_i}^*$. Then $\{\psi_i(x)\}$ is a basis of V_h . Denote by Π_h and Π_h^* the interpolation projectors from $C(\Omega)$ onto U_h and V_h respectively. Then for any $u \in C(\Omega)$

$$\Pi_h u = \sum_{i=1}^N u(P_i) \phi_i(x), \quad \Pi_h^* u = \sum_{i=1}^N u(P_i) \psi_i(x).$$

It is clear that $\Pi_h^* \phi_i(x) = \psi_i(x)$ and consequently $V_h = \text{span}\{\Pi_h^* \phi_h : \phi_h \in U_h\}$.

Set $\Lambda_i = \{j : P_j \text{ is a neighbouring node of } P_i\}$. For adjacent nodes P_i and P_j , write $\Gamma_{ij} = \partial K_{P_i}^* \cap \partial K_{P_j}^*$, and denote by γ_{ij} the length of Γ_{ij} and by ν_{ij} the unit outer normal direction of Γ_{ij} (viewing Γ_{ij} as a part of the boundary of $K_{P_i}^*$). Define

$$\beta_{ij} = \int_{\Gamma_{ij}} b(x) \cdot \nu_{ij} ds. \quad (7.2.3)$$

Then we can divide $\partial K_{P_i}^*$ into a *flow in* part and a *flow out* part according to the sign of β_{ij} :

$$\begin{cases} (\partial K_{P_i}^*)_- = \bigcup_{\substack{\beta_{ij} \leq 0 \\ j \in \Lambda_i}} \Gamma_{ij} & (\text{Flow in}), \\ (\partial K_{P_i}^*)_+ = \bigcup_{\substack{\beta_{ij} > 0 \\ j \in \Lambda_i}} \Gamma_{ij} & (\text{Flow out}). \end{cases} \quad (7.2.4)$$

The following facts are apparent

$$\beta_{ij} + \beta_{ji} = 0, \quad (7.2.5)$$

$$|\beta_{ij}| \leq C \|b\|_{\infty} \gamma_{ij}. \quad (7.2.6)$$

Now we try to write (7.2.1) into a weak form. So we multiply (7.2.1a) by $v_h \in V_h$, integrate it on Ω , and apply Green's formula to obtain

$$a(u, v_h) + b(u, v_h) = (f, v_h), \quad (7.2.7)$$

where

$$a(u, v_h) = - \sum_{j=1}^N v_h(P_j) \int_{\partial K_{P_j}^*} \mu \frac{\partial u}{\partial \nu} ds, \quad (7.2.8)$$

$$b(u, v_h) = \sum_{j=1}^N v_h(P_j) \int_{\partial K_{P_j}^*} (b \cdot \nu) u ds - \int_{\Omega} u v_h \operatorname{div} b dx, \quad (7.2.9)$$

where ν is the unit outer normal direction of $\partial K_{P_j}^*$. The key point to construct a generalized upwind scheme lies in how to approximate the line integral of the first term on the right-hand side of (7.2.9). Write

$$\beta_{jl}^+ = \max(\beta_{jl}, 0), \quad \beta_{jl}^- = \max(-\beta_{jl}, 0). \quad (7.2.10)$$

Taking the *upwind* values leads to the following approximation:

$$\int_{\partial K_{P_j}^*} (b \cdot \nu) u ds \approx \sum_{l \in \Lambda_j} \{\beta_{jl}^+ u(P_j) - \beta_{jl}^- u(P_l)\}. \quad (7.2.11)$$

Let us use the following bilinear form to approximate $b(u, v_h)$:

$$b_h(u, v_h) = \sum_{j=1}^N v_h(P_j) \sum_{l \in \Lambda_j} \{\beta_{jl}^+ u(P_j) - \beta_{jl}^- u(P_l)\} - \int_{\Omega} u v_h \operatorname{div} b dx. \quad (7.2.12)$$

Then we have a generalized upwind difference scheme: Find $u_h \in U_h$ such that

$$a(u_h, v_h) + b_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \quad (7.2.13)$$

This is equivalent to

$$\sum_{i=1}^N a(\phi_i, \psi_j) u_i + \sum_{i=1}^N b_h(\phi_i, \psi_j) u_i = (f, \psi_j), \quad 1 \leq j \leq N, \quad (7.2.13)'$$

where $u_i = u_h(P_i)$, and

$$a(\phi_i, \psi_j) = - \int_{\partial K_{P_j}^*} \mu \frac{\partial \phi_i}{\partial \nu} ds, \quad (7.2.14)$$

$$b_h(\phi_i, \psi_j) = \sum_{l \in \Lambda_j} \{\beta_{ji}^+ \delta_{il} - \beta_{ji}^- \delta_{il}\} - \int_{K_{P_j}^*} \phi_i \operatorname{div} b dx, \quad (7.2.15)$$

where δ_{ij} is the Kronecker delta.

7.2.2 Convergence and error estimate

Write the error as

$$u - u_h = (u - \Pi_h u) + (\Pi_h u - u_h). \quad (7.2.16)$$

The first term on the right-hand side is the error of a linear interpolation, satisfying

$$\|u - \Pi_h u\|_1 \leq Ch \|u\|_2. \quad (7.2.17)$$

To estimate the second term, we note that by (3.2.24) and (3.2.46)

$$a(\bar{u}_h, \Pi_h^* \bar{u}_h) \geq \alpha \|\bar{u}_h\|_1^2, \quad \forall \bar{u}_h \in U_h, \quad \alpha > 0, \quad (7.2.18)$$

$$|a(u - \Pi_h u, \Pi_h^* \bar{u}_h)| \leq Ch \|u\|_2 \|\bar{u}_h\|_1, \quad \bar{u}_h \in U_h. \quad (7.2.19)$$

Next we turn to the estimation of $b_h(u_h, v_h)$. First we rewrite it as

$$b_h(u_h, v_h) = \sum_{j=1}^N \int_{\partial K_{P_j}^*} (\Pi_h^* u_h)^+ v_h (b \cdot \nu_j) ds - \int_{K_{P_j}^*} u_h v_h \operatorname{div} b dx, \quad (7.2.20)$$

where ν_j is the unit outer normal vector, $\Pi_h^* u_h \in V_h$, and $(\Pi_h^* u_h)^+$ is the upwind value of $\Pi_h^* u_h \in V_h$ across the boundary $\partial K_{P_j}^*$. As in §6.2, we may further express $b_h(u_h, v_h)$ as

$$\begin{aligned} b_h(u_h, v_h) = & \sum_{j=1}^N \left\{ \int_{K_{P_j}^*} (b \cdot \nabla \Pi_h^* u_h) v_h dx + \int_{(\partial K_{P_j}^*)^-} [\Pi_h^* u_h] v_h (b \cdot \nu_j) ds \right\} \\ & - \int_{K_{P_j}^*} u_h v_h \operatorname{div} b dx, \end{aligned} \quad (7.2.21)$$

where $[]$ denotes the jump value across the boundary, and we note that $\nabla \Pi_h^* u_h = 0$ on $K_{P_j}^*$. Comparing it with (6.2.8a), we find that $b_h(u_h, v_h)$ here turns out to be $a(u_h, v_h)$ there. So actually, the upwind difference scheme (7.2.13) uses the generalized difference method to discretize the diffusion term, and the discontinuous finite element method to discretize the convection term in (7.2.1a). If we assume

$$-\operatorname{div} b(x) \geq \sigma_0 > 0, \quad (7.2.22)$$

then it follows from (6.2.15) that

$$a(\bar{u}_h, \Pi_h^* \bar{u}_h) + b_h(\bar{u}_h, \Pi_h^* \bar{u}_h) \geq \alpha \|\bar{u}_h\|_1^2, \quad \forall \bar{u}_h \in U_h. \quad (7.2.23)$$

Notice

$$(u - \Pi_h u)(P) = 0, \text{ as } P = P_j, P_l.$$

Thus by (7.2.12) we have

$$b_h(u - \Pi_h u, \Pi_h^* \bar{u}_h) = - \int_{\Omega} (u - \Pi_h u) \Pi_h^* \bar{u}_h \operatorname{div} b dx.$$

This results in the following estimate

$$|b_h(u - \Pi_h u, \Pi_h^* \bar{u}_h)| \leq Ch^2 |u|_2 \|\bar{u}_h\|_0. \quad (7.2.24)$$

Now let u and u_h be the solutions to (7.2.7) and (7.2.13) respectively. Then the subtraction of these two equations leads to an error equation:

$$\begin{aligned} & a(u - u_h, \Pi_h^* \bar{u}_h) + b_h(u - u_h, \Pi_h^* \bar{u}_h) \\ = & -(b(u, \Pi_h^* \bar{u}_h) - b_h(u, \Pi_h^* \bar{u}_h)). \end{aligned} \quad (7.2.25)$$

It remains to show the error order of the right-hand side. To this end, define a function

$$H(x) = \begin{cases} 0, & \text{as } x < 0, \\ 1, & \text{as } x \geq 0. \end{cases}$$

Then we have

$$\begin{aligned} & \beta_{jl}^+ u(P_j) - \beta_{jl}^- u(P_l) \\ &= (H(\beta_{jl})u(P_j) + (1 - H(\beta_{jl}))u(P_l))\beta_{jl} \\ &= \int_{\Gamma_{jl}} b \cdot \nu (H(\beta_{jl})u(P_j) + (1 - H(\beta_{jl}))u(P_l)) ds. \end{aligned}$$

Since $\Gamma_{jl} = \partial K_{P_j}^* \cap \partial K_{P_l}^*$, the integral along Γ_{jl} in the summation on the right-hand side of (7.2.12) appears twice with opposite normal directions ν . Write such two terms together to obtain

$$(v_h(P_j) - v_h(P_l)) \int_{\Gamma_{jl}} b \cdot \nu \{H(\beta_{jl})u(P_j) + (1 - H(\beta_{jl}))u(P_l)\} ds.$$

So we have

$$\begin{aligned} & b_h(u, v_h) \\ &= \frac{1}{2} \sum_{j=1}^N \sum_{l \in \Lambda_j} (v_h(P_j) - v_h(P_l)) \\ & \quad \cdot \int_{\Gamma_{jl}} b \cdot \nu \{H(\beta_{jl})u(P_j) + (1 - H(\beta_{jl}))u(P_l)\} ds \\ & \quad - \int_{\Omega} uv_h \operatorname{div} b dx. \end{aligned} \tag{7.2.26a}$$

Similarly by (7.2.9) we have

$$\begin{aligned} b(u, v_h) &= \frac{1}{2} \sum_{j=1}^N \sum_{l \in \Lambda_j} (v_h(P_j) - v_h(P_l)) \int_{\Gamma_{jl}} b \cdot \nu u ds \\ & \quad - \int_{\Omega} uv_h \operatorname{div} b dx. \end{aligned} \tag{7.2.26b}$$

Subtracting these two equations yields

$$\begin{aligned} & b(u, v_h) - b_h(u, v_h) \\ &= \frac{1}{2} \sum_{j=1}^N \sum_{l \in \Lambda_j} (v_h(P_j) - v_h(P_l)) \int_{\Gamma_{jl}} b \cdot \nu \{ H(\beta_{jl})(u - u(P_j)) \\ & \quad + (1 - H(\beta_{jl}))(u - u(P_l)) \} ds. \end{aligned}$$

As in the deduction of (3.2.46) one can show that

$$\begin{aligned} & |b(u, \Pi_h^* \bar{u}_h) - b_h(u, \Pi_h^* \bar{u}_h)| \\ & \leq C \|b\|_\infty h |u|_2 |\bar{u}_h|_1, \quad \forall \bar{u}_h \in U_h. \end{aligned} \quad (7.2.27)$$

By the positive definiteness (7.2.23), the continuity (7.2.19), the consistency (7.2.27), and as in the error estimation in §3.2, one can easily prove the following theorem.

Theorem 7.2.1 *Assume that b satisfies (7.2.22), that $b \in H^1(\Omega) \times H^1(\Omega) \cap L^\infty(\Omega) \times L^\infty(\Omega)$, that $f \in L^2(\Omega)$, and that the solution u of (7.2.1) belongs to $H^2(\Omega) \cap H_0^1(\Omega)$. Then, there holds the following error estimate:*

$$\|u - u_h\|_1 \leq Ch |u|_2.$$

Remark The above theorem requires that b satisfies (7.2.22). But this is not an essential restriction. In fact, if necessary, we can always validate (7.2.22) by performing the following transformation

$$\bar{u} = u e^{\omega t}, \quad \omega = \sigma_0 + \frac{1}{2} |\operatorname{div} b|.$$

7.2.3 Extreme value theorem and uniform convergence

First let us have a look at the signs of the coefficients in the generalized difference equation (7.2.13). In Fig 7.2.3, $\triangle P_i P_j P_k$ and $\triangle P_i P_{k'} P_j$ are two adjacent triangle elements, and Q and Q' are their barycenters (or circumcenters) respectively. Write $K_Q = \triangle P_i P_j P_k$ and $K_{Q'} = \triangle P_i P_{k'} P_j$. Let m_i , m_j , and m_k be the three midpoints

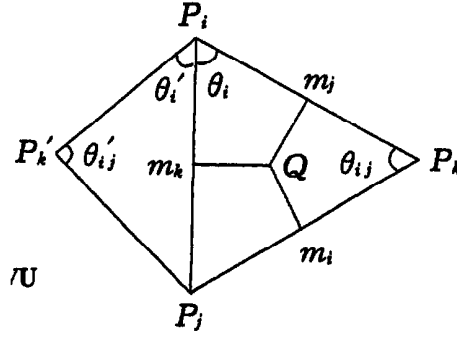


Fig. 7.2.3

of the sides of K_Q , and let θ_{ij} , θ'_{ij} , θ_i and θ'_i be the inner angles of the elements.

Let us try to evaluate

$$a(\phi_i, \psi_j) = - \int_{\partial K_{P_j}^*} \mu \frac{\partial \phi_i}{\partial \nu} ds.$$

If $i \notin \Lambda_j$, then it is obvious that

$$a(\phi_i, \psi_j) = 0.$$

On the other hand, if $i \in \Lambda_j$ and $i \neq j$, then

$$a(\phi_i, \psi_j) = - \int_{\partial K_{P_j}^* \cap K_Q} \mu \frac{\partial \phi_i}{\partial \nu} ds - \int_{\partial K_{P_j}^* \cap K_{Q'}} \mu \frac{\partial \phi_i}{\partial \nu} ds.$$

Apply Green's formula on $\Delta m_i Q m_k$ to get

$$- \int_{\partial K_{P_j}^* \cap K_Q} \mu \frac{\partial \phi_i}{\partial \nu} ds = \int_{m_k m_i} \mu \frac{\partial \phi_i}{\partial \nu} ds = \int_{m_k m_i} \mu \nabla \phi_i \nu ds.$$

Note that ϕ_i is actually an area coordinate on $\Delta P_i P_j P_k$. Hence according to the expression of area coordinates by rectangular coordinates, the length of $\nabla \phi_i$ is

$$|\overline{P_j P_k}| / (2S_Q), \quad (S_Q \text{ being the area of } K_Q)$$

and it is perpendicular to $\overline{P_j P_k}$, pointing towards P_i . ν is a unit vector perpendicular to $\overline{m_i m_k}$ (i.e. to $\overline{P_i P_k}$) and pointing towards P_j . Therefore we have

$$-\int_{\partial K_{P_j}^* \cap K_Q} \mu \frac{\partial \phi_i}{\partial \nu} ds = -\frac{\mu |\overline{P_j P_k}| |\overline{P_k P_i}|}{4S_Q} \cos \theta_{ij} = -\frac{\mu}{2} \operatorname{ctg} \theta_{ij}.$$

Similarly

$$-\int_{\partial K_{P_j}^* \cap K_{Q'}} \mu \frac{\partial \phi_i}{\partial \nu} ds = -\frac{\mu}{2} \operatorname{ctg} \theta'_{ij}.$$

So, for $i \in \Lambda_j$ but $i \neq j$, we have

$$a(\phi_i, \psi_j) = -\frac{\mu}{2} (\operatorname{ctg} \theta_{ij} + \operatorname{ctg} \theta'_{ij}). \quad (7.2.28)$$

In the case of $i = j$, i.e. $\phi_i = \phi_j$, we use Green's formula on the quadrilateral $P_j m_i Q m_k$ to obtain

$$\begin{aligned} & -\int_{\partial K_{P_j}^* \cap K_Q} \mu \frac{\partial \phi_j}{\partial \nu} ds \\ &= \int_{P_j m_i} \mu \frac{\partial \phi_j}{\partial \nu} ds + \int_{m_k P_j} \mu \frac{\partial \phi_j}{\partial \nu} ds \\ &= \frac{\mu |\overline{P_k P_i}| |\overline{P_k P_j}|}{4S_Q} \cos \theta_{ij} + \frac{\mu |\overline{P_k P_i}| |\overline{P_i P_j}|}{4S_Q} \cos \theta_i \\ &= \frac{\mu}{2} (\operatorname{ctg} \theta_{ij} + \operatorname{ctg} \theta_i). \end{aligned}$$

Recall that θ_{ij} and θ_i are two inner angles of the element K_Q , of which the vertexes are not P_j . Thus we have

$$a(\phi_j, \psi_j) = -\int_{\partial K_{P_j}^*} \mu \frac{\partial \phi_j}{\partial \nu} ds = \frac{\mu}{2} \sum_{i \in \Lambda_j} (\operatorname{ctg} \theta_i + \operatorname{ctg} \theta'_i). \quad (7.2.29)$$

Now we assume T_h is an acute triangulation, namely there exists a constant $\epsilon_0 > 0$ such that any inner angle θ of an element of T_h satisfies $\theta \leq \frac{\pi}{2} - \epsilon_0$. Then none of the angles appearing in (7.2.28) and (7.2.29) is greater than $\frac{\pi}{2} - \epsilon_0$, and hence

$$\begin{cases} a(\phi_i, \phi_j) \leq 0, & \text{when } i \neq j, \\ a(\phi_i, \phi_j) \geq \delta_0 > 0, & \text{when } i = j. \end{cases} \quad (7.2.30)$$

To evaluate $b_h(\phi_i, \psi_j)$ we first note that by (7.2.15)

$$b_h(\phi_i, \psi_j) = 0, \text{ when } i \text{ and } j \text{ are not adjacent,} \quad (7.2.31)$$

$$b_h(\phi_j, \psi_j) = \sum_{l \in \Lambda_j} \beta_{jl}^+ - \int_{K_{P_j}^*} \phi_j \operatorname{div} b dx \geq 0. \quad (7.2.32)$$

When i and j are adjacent but not identical we have

$$\begin{aligned} b_h(\phi_i, \psi_j) &= -\beta_{ji}^- - \int_{K_{P_j}^*} \phi_i \operatorname{div} b dx \\ &= - \int_{(\partial K_{P_j}^*)^-} b \cdot \nu ds - \int_{K_{P_j}^*} \phi_i \operatorname{div} b dx. \end{aligned}$$

Notice that for a quasi-uniform grid T_h , the area $S_{P_j}^*$ of $K_{P_j}^*$ is of order h^2 , that is, there exist positive constants α_1 and α_2 such that $\alpha_1 h^2 \leq S_{P_j}^* \leq \alpha_2 h^2$. Thus for sufficiently small h

$$0 \leq b_h(\phi_i, \psi_j) \leq \alpha_0 h \quad (\alpha_0 > 0). \quad (7.2.33)$$

We also observe that by (7.2.15) and Green's formula

$$\sum_{i=1}^N b_h(\phi_i, \psi_j) = \sum_{l \in \Lambda_i} \beta_{jl} - \int_{K_{P_j}^*} \operatorname{div} b dx = 0. \quad (7.2.34)$$

Now we are ready to study the extreme value property of (7.2.13) or (7.2.13)'. Set

$$a_{ij} = a(\phi_i, \psi_j) + b_h(\phi_i, \psi_j), \quad b_j = (f, \psi_j).$$

Write (7.2.13)' in the form

$$\sum_{i=1}^N a_{ij} u_i = b_j, \quad j = 1, 2, \dots, N.$$

It follows from (7.2.28)–(7.2.34) that if T_h is an acute triangulation and h is sufficiently small, then $a_{ij} \leq 0$ for $i \neq j$, $a_{jj} > 0$, and

$$\sum_{j=1}^N a_{ij} \geq 0, \quad 1 \leq i \leq N. \quad (7.2.35a)$$

Observe that (7.2.29) remains to be true even if one of the neighbouring nodes of P_j is on the boundary. But (7.2.28) is valid only when $i \in \Lambda_j$ and P_i is not a boundary node. This term vanishes if P_i is indeed a boundary node. These observations imply the existence of $d_0 > 0$ and j_0 such that

$$\sum_{i=1}^N a_{ij_0} \geq d_0. \quad (7.2.35b)$$

Hence, the extreme value theorem of difference equations (cf. [A-27] and [C-7]) gives

$$\max_{1 \leq i \leq N} |u_i| \leq C \max_{1 \leq i \leq N} |b_i|. \quad (7.2.36)$$

Since $u_h(x)$ is a piecewise linear function, furthermore we have (cf. [A-17])

Theorem 7.2.2 (*Extreme value theorem*) Let T_h be an acute triangulation, and let b satisfy the conditions of Theorem 7.2.1, then

$$\|u_h\|_\infty \leq C \max_i |(f, \psi_i)|. \quad (7.2.37)$$

By the properties of the matrix $A = (a_{ij})$ mentioned above, we also know that A^{-1} is a non-negative nonsingular matrix ([B-93]). So if the right-hand side term f and the boundary value g are non-negative, then the difference solution is non-negative as well. Thus the difference solution will not have unnecessary oscillations.

Next let us turn to the estimation in maximum norm. Let u and u_h be the solutions to the convection-diffusion equation and the difference equation respectively. Use the triangular inequality to get

$$\|u - u_h\|_\infty \leq \|u - \Pi_h u\|_\infty + \|\Pi_h u - u_h\|_\infty. \quad (7.2.38)$$

Take a fixed $p > 2$ (2 being the dimension of the plane), then by virtue of the Sobolev imbedding theorem we have

$$\|u - \Pi_h u\|_\infty \leq C \|u - \Pi_h u\|_{1,p}.$$

It follows from the interpolation approximation theory in Sobolev spaces ([B-17]) that

$$\|u - \Pi_h u\|_{1,p} \leq Ch |u|_{2,p}.$$

Thus

$$\|u - \Pi_h u\|_\infty \leq Ch|u|_{2,p}. \quad (7.2.39)$$

To estimate the second term of the right-hand side of (7.2.38), we notice that $\Pi_h u - u_h$ satisfies the following equation

$$a(\Pi_h u - u_h, \Pi_h^* \phi_j) + b_h(\Pi_h u - u_h, \Pi_h^* \phi_j) = r_j,$$

where

$$\begin{aligned} r_j = & a(\Pi_h u - u, \Pi_h^* \phi_j) + b_h(\Pi_h u - u, \Pi_h^* \phi_j) \\ & - (b(u, \Pi_h^* \phi_j) - b_h(u, \Pi_h^* \phi_j)). \end{aligned}$$

A combination of (7.2.19), (7.2.24) and (7.2.26) leads to an estimation of r_j :

$$|r_j| \leq Ch|u|_2.$$

Thus in terms of the extreme value property (7.2.37) we have

$$\|\Pi_h u - u_h\|_\infty \leq Ch|u|_2. \quad (7.2.40)$$

Connecting (7.2.38)–(7.2.40) yields (cf. [A-17])

Theorem 7.2.3 *Under the assumptions of Theorem 7.2.2, the following maximum estimate holds for any $p > 2$*

$$\|u - u_h\|_\infty \leq Ch|u|_{2,p}. \quad (7.2.41)$$

7.2.4 Mass conservation

Consider a conservative equation:

$$\begin{cases} -\mu \Delta u + \nabla(bu) = f, & \text{in } \Omega, \end{cases} \quad (7.2.42a)$$

$$\begin{cases} u = 0, & \text{on } \partial\Omega. \end{cases} \quad (7.2.42b)$$

The corresponding generalized upwind equation is

$$a(u_h, \psi_j) + b_h(u_h, \psi_j) = (f, \psi_j), \quad (7.2.43)$$

where

$$a(u_h, \psi_j) = - \int_{\partial K_{P_j}^*} \mu \frac{\partial u_h}{\partial \nu} ds,$$

$$b_h(u_h, \psi_j) = \sum_{l \in \Lambda_j} [\beta_{jl}^+ u_h(P_j) - \beta_{jl}^- u_h(P_l)].$$

Integrate (7.2.42a) on Ω and exploit Green's formula to obtain

$$\int_{\partial\Omega} (-\mu \frac{\partial u}{\partial \nu} + b \cdot \nu u) ds = \int_{\Omega} f dx. \quad (7.2.44)$$

Here the left-hand side is the mass "flowing out" of Ω through the boundary $\partial\Omega$, and the right-hand side the mass out of the source f . Therefore, the equation (7.2.44) describes a mass conservation. Next, finding the sum of the difference equation (7.2.43) gives

$$\begin{aligned} & - \sum_{j=1}^N \int_{\partial K_{P_j}^*} \mu \frac{\partial u_h}{\partial \nu} ds + \sum_{j=1}^N \int_{\Gamma_{jl}} b \cdot \nu [H(\beta_{jl}) u_j + (1 - H(\beta_{jl})) u_l] ds \\ & = \int_{\Omega_h^*} f dx, \end{aligned}$$

where $\Omega_h^* = \bigcup_{j=1}^N K_{P_j}^*$. Obviously the left-hand side terms cancel each other on the inner boundaries, so the above equation becomes:

$$\begin{aligned} & - \int_{\partial\Omega_h^*} \mu \frac{\partial u_h}{\partial \nu} ds + \sum_{\Gamma_{ij} \in \partial\Omega_h^*} \int_{\Gamma_{ij}} b \cdot \nu [H(\beta_{ij}) u_i + (1 - H(\beta_{ij})) u_j] ds \\ & = \int_{\Omega_h^*} f dx. \end{aligned} \quad (7.2.45)$$

This is precisely a discrete mass conservation law.

7.3 Generalized Upwind Difference Schemes for Non-steady-state Problems

In this and the next sections we discuss generalized upwind difference solutions of the following non-steady-state convection-diffusion problem:

$$\begin{cases} \frac{\partial u}{\partial t} = \mu \Delta u - b \cdot \nabla u + f, & x \in \Omega, 0 < t \leq T, \\ u(x, t) = 0, & x \in \Gamma = \partial\Omega, 0 < t \leq T, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad \begin{matrix} (7.3.1a) \\ (7.3.1b) \\ (7.3.1c) \end{matrix}$$

where $x = (x_1, x_2)$, $\mu > 0$ is the diffusion coefficient, $b = (b_1(x), b_2(x))$ is the convection speed. Usually $u = u(x, t)$ stands for the density or the temperature, and $f(x)$ some kind of source.

7.3.1 Construction of difference schemes

As in §7.2, we place a triangulation T_h and a dual grid T_h^* on Ω , introduce a trial function space U_h and a test function space V_h , and define the following bilinear forms:

$$a(u, v_h) = - \sum_{j=1}^N v_h(P_j) \int_{\partial K_{P_j}^*} \mu \frac{\partial u}{\partial \nu} ds, \quad (7.3.2)$$

$$b(u, v_h) = \sum_{j=1}^N v_h(P_j) \int_{\partial K_{P_j}^*} (b \cdot \nu) u ds - \int_{\Omega} u v_h \operatorname{div} b dx, \quad (7.3.3)$$

$$b_h(u, v_h) = \sum_{j=1}^N v_h(P_j) \sum_{l \in \Lambda_j} \{\beta_{jl}^+ u(P_j) - \beta_{jl}^- u(P_l)\} - \int_{\Omega} u v_h \operatorname{div} b dx, \quad (7.3.4)$$

where

$$\beta_{jl}^+ = \max(\beta_{jl}, 0), \quad \beta_{jl}^- = \max(-\beta_{jl}, 0), \quad (7.3.5)$$

$$\beta_{ij} = \int_{\Gamma_{ij}} b(x) \cdot \nu_{ij} ds. \quad (7.3.6)$$

A weak form of (7.3.1) is: Find $u \in C^1([0, T]; H_0^1(\Omega))$ such that

$$\left(\frac{\partial u}{\partial t}, v \right) + a(u, v) + b(u, v) = (f, v), \quad v \in H_0^1(\Omega). \quad (7.3.7)$$

A semi-discrete generalized upwind difference scheme for (7.3.7) is: Find $u_h(\cdot, t) \in U_h$ such that

$$\left(\frac{\partial u_h}{\partial t}, v_h \right) + a(u_h, v_h) + b_h(u_h, v_h) = (f, v_h), \quad v_h \in V_h. \quad (7.3.8)$$

Choose a time step size $\tau = T/N$ and the nodes $t_k = k\tau$ ($k = 0, 1, \dots, N$). A fully-discrete, explicit, generalized upwind difference scheme is: Find $u_h^k \in U_h$ such that

$$\begin{aligned} ((u_h^k - u_h^{k-1})/\tau, v_h) + a(u_h^{k-1}, v_h) + b_h(u_h^{k-1}, v_h) &= (f, v_h), \\ \forall v_h \in V_h. \end{aligned} \quad (7.3.9)$$

In particular, choose v_h as a basis function ψ_j of V_h , and set

$$\bar{\partial}_t u_h^k = (u_h^k - u_h^{k-1})/\tau.$$

Then we have (cf. Figs. 7.2.1 and 7.2.2)

$$\begin{aligned} \int_{K_{P_j}^*} \bar{\partial}_t u_h^k dx &= \sum_{l \in \Lambda_j} \mu \frac{u_{jl}^{k-1} - u_j^{k-1}}{|P_j P_{jl}|} \gamma_{jl} - \sum_{l \in \Lambda_j} \{\beta_{jl}^+ u_j^{k-1} - \beta_{jl}^- u_l^{k-1}\} \\ &\quad - \int_{K_{P_j}^*} u_h^{k-1} \operatorname{div} b dx + \int_{K_{P_j}^*} f dx, \end{aligned} \quad (7.3.10)$$

where γ_{jl} is the length of Γ_{jl} . The initial value u_h^0 satisfies

$$(u_h^0, \psi_j) = (u_0, \psi_j). \quad (7.3.11)$$

Let us introduce the symbol $u_h^{k,\theta} = \theta u_h^k + (1-\theta)u_h^{k-1}$ ($0 \leq \theta \leq 1$), then we can define a more general weighted scheme:

$$\begin{aligned} (\bar{\partial}_t u_h^k, v_h) + a(u_h^{k,\theta}, v_h) + b_h(u_h^{k,\theta}, v_h) &= (f, v_h), \\ \forall v_h \in V_h. \end{aligned} \quad (7.3.12)$$

This corresponds to a backward difference scheme if $\theta = 1$ and a Crank-Nicolson scheme if $\theta = \frac{1}{2}$. We remark that the difference equation (7.1.8) now can be written as

$$\begin{aligned} (\bar{\partial}_t u_h^k, v_h) + a(u_h^k, v_h) + b_h(u_h^{k-1}, v_h) &= (f, v_h), \\ \forall v_h \in V_h. \end{aligned} \quad (7.3.13)$$

Remark 1 Baba and Tabata ([B-3]) proposed a upwind finite element scheme, which can be written in our symbols as

$$(\bar{\partial}_t u_h^k, \Pi_h^* \bar{u}_h) + a(u_h^{k,\theta}, \bar{u}_h) + b_h(u_h^{k,\theta}, \Pi_h^* \bar{u}_h) = (f, \Pi_h^* \bar{u}_h),$$

$$\forall \bar{u}_h \in U_h. \quad (7.3.14)$$

Remark 2 In order to simplify the computation of the difference schemes, one often replaces $\bar{\partial}_t$ by $\Pi_h^* \bar{\partial}_t$, and $(\bar{\partial}_t u_h^{k-1}, v_h)$ by $(\Pi_h^* \bar{\partial}_t u_h^{k-1}, v_h)$. In such a case, we have

$$(\Pi_h^* \bar{\partial}_t u_h^{k-1}, \psi_j) = S_{P_j}^* (u_j^k - u_j^{k-1}) / \tau,$$

where $S_{P_j}^*$ is the area of $K_{P_j}^*$.

7.3.2 Convergence and error estimate

In this subsection, we consider the case that T_h^* is a barycenter dual grid. Taking $v_h = \Pi_h^* u_h$ in the semi-discrete scheme (7.3.8) yields

$$\frac{\partial}{\partial t} |||u_h|||_0^2 + a(u_h, \Pi_h^* u_h) + b_h(u_h, \Pi_h^* u_h) = (f, \Pi_h^* u_h), \quad (7.3.15)$$

where (cf §5.1)

$$|||u_h|||_0^2 = (u_h, \Pi_h^* u_h). \quad (7.3.16)$$

By §3.2, there is a constant $\alpha > 0$ such that

$$a(u_h, \Pi_h^* u_h) \geq \alpha |u_h|_1^2. \quad (7.3.17)$$

It follows from (7.2.26a) that

$$\begin{aligned} b_h(u_h, \Pi_h^* \bar{u}_h) &= \frac{1}{2} \sum_{j=1}^N \sum_{l \in \Lambda_j} (\bar{u}_h(P_j) - \bar{u}_h(P_l)) \cdot \\ &\quad \cdot \int_{\Gamma_{jl}} b \cdot \nu \{H(\beta_{jl})u_h(P_j) + (1 - H(\beta_{jl}))u_h(P_l)\} ds \\ &\quad - \int_{\Omega} u_h \Pi_h^* \bar{u}_h \operatorname{div} b dx. \end{aligned}$$

Evidently

$$\begin{aligned} &|b_h(u_h, \Pi_h^* \bar{u}_h)| \\ &\leq C_1 \|b\|_{\infty} |\bar{u}_h|_1 \|u_h\|_0 + C_2 \|\operatorname{div} b\|_{\infty} \|\bar{u}_h\|_0 \|u_h\|_0. \end{aligned} \quad (7.3.18)$$

Here we have used the equivalence of $\|\Pi_h^* \bar{u}_h\|_0$ and $\|\bar{u}_h\|_0$ (cf. §5.1). By virtue of (7.3.15), (7.3.17), (7.3.18) and the ϵ -inequality we have

$$\frac{\partial}{\partial t} \|u_h\|_0^2 + \alpha_0 |u_h|_1^2 \leq C(\|u_h\|_0^2 + \|f\|_0^2), \quad (7.3.19)$$

where $\alpha_0 > 0$ is a constant. Note that $\|u_h\|_0$ and $\|u_h\|_0$ are equivalent, view $\phi(t) = \|u_h\|_0$ as a unknown function, and integrate the above inequality, then we have

$$\begin{aligned} & \|u_h(t)\|_0^2 + \alpha_0 \int_0^t |u_h|^2 dt \\ & \leq C(\|u_h^0\|_0^2 + \int_0^t \|f\|_0^2 dt), \quad 0 \leq t \leq T. \end{aligned} \quad (7.3.20)$$

Therefore the semi-discrete solution $u_h(t)$ is stable respect to the initial value and the right-hand side.

Now we turn to deal with the error of the semi-discretization. Write

$$\begin{aligned} u - u_h &= \rho_h + e_h, \\ \rho_h &= u - \Pi_h u, \quad e_h = \Pi_h u - u_h, \end{aligned} \quad (7.3.21)$$

Then e_h satisfies

$$\begin{aligned} & \left(\frac{\partial e_h}{\partial t}, v_h \right) + a(e_h, v_h) + b_h(e_h, v_h) \\ &= (\Pi_h u_t - u_t, v_h) - a(\rho_h, v_h) + (b_h(\Pi_h u, v_h) - b(u, v_h)). \end{aligned} \quad (7.3.22)$$

Take $v_h = \Pi_h^* e_h$, and employ (7.2.19), (7.2.24), (7.2.27) and the ϵ -inequality, then we obtain an inequality similar to (7.3.19):

$$\frac{\partial}{\partial t} \|e_h\|_0^2 + \alpha_0 |e_h|_1^2 \leq C(\|e_h\|_0^2 + h^2 |u|_2^2 + h^2 |u_t|_1^2).$$

Integrating the above inequality leads to another inequality analogous to (7.3.20):

$$\begin{aligned} & \|e_h(t)\|_0^2 + \alpha_0 \int_0^t |e_h(t)|_1^2 dt \\ & \leq Ch^2 \left(\|e_h^0\|_0^2 + \int_0^t |u(t)|_2^2 dt + \int_0^t |u_t(t)|_1^2 dt \right). \end{aligned} \quad (7.3.23)$$

Connect (7.3.21)-(7.2.23) and note $e_h^0 = 0$, then we have the following error estimate for the semi-discrete solution:

$$\begin{aligned} & \|u - u_h\|_0^2 + \int_0^t |u - u_h|_1^2 dt \\ & \leq Ch^2 \left(\|u\|_2^2 + \int_0^t |u|_2^2 dt + \int_0^t |u_t|_1^2 dt \right). \end{aligned} \quad (7.3.24)$$

Next we turn to deal with the fully-discrete scheme. Choose $\theta = 1$ in (7.3.12) to obtain

$$\begin{aligned} & ((u_h^k - u_h^{k-1})\tau^{-1}, v_h) + a(u_h^k, v_h) + b_h(u_h^k, v_h) \\ & = (f, v_h), \quad \forall v_h \in V_h. \end{aligned}$$

As before, we only need to estimate $e_h^k = \Pi_h u^k - u_h^k$, which satisfies the equation

$$\begin{aligned} & (\bar{\partial}_t e_h^k, v_h) + a(e_h^k, v_h) + b_h(e_h^k, v_h) \\ & = (\Pi_h u_t^k - u_t^k, v_h) + (\Pi_h(\bar{\partial}_t u^k - u_t^k), v_h) \\ & \quad + a(-\rho_h^k, v_h) + (b_h(\Pi_h u^k, v_h) - b(u^k, v_h)) \\ & = R_{1h}^k + R_{2h}^k + R_{3h}^k + R_{4h}^k. \end{aligned} \quad (7.3.25)$$

Let $v_h = \Pi_h^* e_h^k$. Then

$$|R_{1h}^k| \leq Ch^2 |u_t^k|_2 \|e_h^k\|_0 \leq Ch^2 (\|e_h^k\|_0^2 + |u_t^k|_2^2), \quad (7.3.26a)$$

$$|R_{2h}^k| \leq C\tau \|u_{tt}^k\|_0 \|e_h^k\|_0 \leq Ch^2 (\|e_h^k\|_0^2 + \|u_{tt}^k\|_0^2), \quad (7.3.26b)$$

$$|R_{3h}^k| \leq Ch |u^k|_2 |e_h^k|_1 \leq C(\epsilon^2 |e_h^k|_1^2 + \epsilon^{-2} h^2 |u^k|_2^2). \quad (7.3.26c)$$

It follows from (7.2.27) that

$$|R_{4h}^k| \leq Ch |u^k|_2 |e_h^k|_1 \leq C(\epsilon^2 |e_h^k|_1^2 + \epsilon^{-2} h^2 |u^k|_2^2). \quad (7.3.26d)$$

To estimate the left-hand side of (7.3.25), we note

$$\begin{aligned} & \frac{1}{\tau} (e_h^k, \Pi_h^* e_h^k) - \frac{1}{\tau} (e_h^{k-1}, \Pi_h^* e_h^k) + a(e_h^k, \Pi_h^* e_h^k) + b_h(e_h^k, \Pi_h^* e_h^k) \\ & \geq \frac{1}{\tau} (|||e_h^k|||_0^2 - |||e_h^{k-1}|||_0^2) + \alpha_0 |e_h^k|_1^2 \\ & \geq \frac{1}{2\tau} (|||e_h^k|||_0^2 - |||e_h^{k-1}|||_0^2) + \alpha_0 |e_h^k|_1^2. \end{aligned}$$

Choose ϵ sufficiently small such that the sum of the coefficients of $|e_h^k|_1^2$ in (7.3.26c,d) is less than α_0 , then there exists an $\alpha_1 > 0$ such that

$$\begin{aligned} & \frac{1}{2\tau} (|||e_h^k|||_0^2 - |||e_h^{k-1}|||_0^2) + \alpha_1 |e_h^k|_1^2 \\ & \leq Ch^2 (\|e_h^k\|_0^2 + |u^k|_2^2 + |u_t^k|_2^2 + \|u_{tt}^k\|_0^2). \end{aligned} \quad (7.3.27)$$

Find the sum with respect to k , notice the equivalence of $|||e_h^k|||_0$ and $\|e_h^k\|_0$, and note $k\tau \leq T$ and $e_h^0 = 0$, then we have

Theorem 7.3.1 *Suppose $u \in C^2([0, T]; H^2(\Omega))$, then the backward difference solution $\{u_h^n\}$ satisfies the following error estimate:*

$$\max_{1 \leq k \leq N} \|u(t_k) - u_h^k\|_0 \leq Ch \|u\|_{X_1}, \quad (7.3.28)$$

where

$$\begin{cases} X_1 = C^1([0, T]; H^2(\Omega)) \cap C^2([0, T]; L^2(\Omega)), \\ \|u\|_{X_1} = \|u\|_{C^1([0, T]; H^2(\Omega))} + \|u\|_{C^2([0, T]; L^2(\Omega))}. \end{cases} \quad (7.3.29)$$

7.4 Highly Accurate Generalized Upwind Schemes

The generalized upwind schemes introduced by now are all of first order accuracy. In this section we combine generalized difference methods with higher order upwind schemes in Chapter 6 to construct a class of generalized upwind schemes for convection-dominated diffusion equations, which, in principle, can reach arbitrarily high order accuracy.

7.4.1 Construction of the difference schemes

Again we try to solve (7.3.1) and keep all the assumptions on the coefficients and the solution region there. As in §7.2, we assume T_h is a quasi-uniform triangulation, and T_h^* a barycenter or circumcenter dual grid. Here and below, the meanings of the symbols are the same as in the last section unless otherwise stated. For simplicity we only

construct linear element scheme. Extensions to higher order elements are self-evident.

As before we construct a trial function space

$$U_h = \{u_h(x) \in C(\bar{\Omega}) : u_h(x)|_K \text{ is linear } \forall K \in T_h \text{ and } u_h|_{\Gamma} = 0\}.$$

For each inner node P_i , there is a basis function $\phi_i(x)$ of U_h , which equals to 1 at P_i (cf. Figg. 7.2.1, 7.2.2), and equals to 0 at other nodes. A function $u_h \in U_h$ has the expression

$$u_h(x) = \sum_{i=1}^N u_h(P_i) \phi_i(x).$$

The test function space relative to the dual grid T_h^* is given as

$$V_h = \{v_h(x) : v_h(x) \text{ is piecewise constant on } T_h^*; \\ v_h \text{ vanishes on } K_{P_i}^* \text{ when } P_i \text{ is a boundary node}\}.$$

For $j = 1, 2, \dots, N$, the basis function $\psi_j(x)$ is chosen as the characteristic function of $K_{P_j}^*$. Let Π_h and Π_h^* be the interpolation projectors defined in §7.2. Then for any $u \in C(\bar{\Omega})$ we have

$$\Pi_h u = \sum_{i=1}^N u(P_i) \phi_i(x), \quad \Pi_h^* u = \sum_{j=1}^N u(P_j) \psi_j(x).$$

Let $u_h(\cdot, t) \in U_h$ ($0 < t \leq T$), the approximation solution of (7.3.1), satisfy formally

$$\int_{K_{P_j}^*} \left[\frac{\partial u_h}{\partial t} - \mu \Delta u_h + b \cdot \nabla (\Pi_h^* u_h) \right] v_h dx = \int_{K_{P_j}^*} f v_h dx, \quad v_h \in V_h. \quad (7.4.1)$$

In terms of Green's formula we have

$$-\mu \int_{K_{P_j}^*} \Delta u_h \cdot v_h dx = -\mu \int_{\partial K_{P_j}^*} \frac{\partial u_h}{\partial \nu} v_h ds. \quad (7.4.2)$$

As in Chapter 6, we apply Green's formula to the convection term in the following fashion:

$$\begin{aligned} & \int_{K_{P_j}^*} b \cdot \nabla(\Pi_h^* u_h) v_h dx \\ &= - \int_{K_{P_j}^*} (\Pi_h^* u_h) \operatorname{div}(b v_h) dx + \int_{\partial K_{P_j}^*} b \cdot \nu (\Pi_h^* u_h) v_h ds. \end{aligned} \quad (7.4.3)$$

Denote by $(\Pi_h^* u_h)^+$ and $(\Pi_h^* u_h)^-$ the upwind and downwind values (cf. §6.2) of $\Pi_h^* u_h$ across the boundary $\partial K_{P_j}^*$. Then, it follows from $\nabla(\Pi_h^* u_h) = 0$ and (7.4.3) that

$$\begin{aligned} & \int_{K_{P_j}^*} (\Pi_h^* u_h) \operatorname{div}(b v_h) dx \\ &= \int_{(\partial K_{P_j}^*)^-} (b \cdot \nu) (\Pi_h^* u_h)^- ds + \int_{(\partial K_{P_j}^*)^+} (b \cdot \nu) (\Pi_h^* u_h)^+ ds. \end{aligned} \quad (7.4.4)$$

Next we replace $\Pi_h^* u_h$ in the second term on the right-hand side of (7.4.3) by $(\Pi_h^* u_h)^+$, and substitute (7.4.4) into the first term to obtain

$$\int_{K_{P_j}^*} b \cdot \nabla(\Pi_h^* u_h) v_h dx \approx \int_{(\partial K_{P_j}^*)^-} (b \cdot \nu) [\Pi_h^* u_h] v_h ds, \quad (7.4.5)$$

where

$$[\Pi_h^* u_h] = (\Pi_h^* u_h)^+ - (\Pi_h^* u_h)^-.$$

Finally, inserting (7.4.2) and (7.4.5) into (7.4.1) yields a semi-discrete generalized upwind scheme:

$$\begin{aligned} & \int_{K_{P_j}^*} \frac{\partial u_h}{\partial t} v_h dx \\ &= \mu \int_{\partial K_{P_j}^*} \frac{\partial u_h}{\partial \nu} v_h ds - \int_{(\partial K_{P_j}^*)^-} (b \cdot \nu) [\Pi_h^* u_h] v_h ds + \int_{K_{P_j}^*} f v_h dx. \end{aligned} \quad (7.4.6)$$

Let us introduce the following bilinear forms:

$$a(u, v_h) = - \sum_{j=1}^N v_h(P_j) \int_{\partial K_j^*} \mu \frac{\partial u}{\partial \nu} ds, \quad (7.4.7)$$

$$b(u, v_h) = \sum_{j=1}^N \int_{K_{P_j}^*} (b \cdot \nabla u) v_h dx \quad (7.4.8a)$$

$$= \sum_{j=1}^N v_h(P_j) \int_{\partial K_{P_j}^*} b \cdot \nu u ds - \int_{\Omega} u v_h \operatorname{div} b dx,$$

$$b_h(u_h, v_h) = \sum_{j=1}^N \int_{(\partial K_{P_j}^*)_-} (b \cdot \nu) [\Pi_h^* u_h] v_h ds. \quad (7.4.8b)$$

Then, the solution u to (7.3.1) satisfies

$$\left(\frac{\partial u}{\partial t}, v_h \right) + a(u, v_h) + b(u, v_h) = (f, v_h), \quad \forall v_h \in V_h. \quad (7.4.9)$$

(7.4.6) can be written as

$$\left(\frac{\partial u_h}{\partial t}, v_h \right) + a(u_h, v_h) + b_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \quad (7.4.6)'$$

Remark 1 If we adopt a higher order finite element space U_h , and a corresponding polynomial function space V_h on T_h^* (cf. §3.4 and §3.5), then $\nabla(\Pi_h^* u_h) \neq 0$, and (7.4.8b) should be modified as

$$\begin{aligned} & b_h(u_h, v_h) \\ &= \sum_{j=1}^N \int_{K_{P_j}^*} b \cdot \nabla(\Pi_h^* u_h) v_h dx + \sum_{j=1}^N \int_{(\partial K_{P_j}^*)_-} (b \cdot \nu) [\Pi_h^* u_h] v_h ds. \end{aligned} \quad (7.4.8c)$$

Take a time step size $\tau = T/N$ (N is a positive integer), and the nodes $t_k = k\tau$ ($k = 0, 1, \dots, N$). Use u_h^k for the approximation of $u(x, k\tau)$ and introduce the symbols $u_h^{k,\theta} = \theta u_h^k + (1-\theta)u_h^{k-1}$ ($0 \leq \theta \leq 1$), then a class of fully-discrete generalized upwind schemes approximating (7.3.1) is:

$$(\bar{\partial}_t u_h^k, v_h) + a(u_h^{k,\theta}, v_h) + b_h(u_h^{k,\theta}, v_h) = (f, v_h), \quad \forall v_h \in V_h. \quad (7.4.10)$$

This leads to an explicit forward and an implicit backward scheme as $\theta = 0, 1$ respectively:

$$(\bar{\partial}_t u_h^k, v_h) + a(u_h^{k-1}, v_h) + b_h(u_h^{k-1}, v_h) = (f, v_h), \quad (7.4.11)$$

$$(\bar{\partial}_t u_h^k, v_h) + a(u_h^k, v_h) + b_h(u_h^k, v_h) = (f, v_h). \quad (7.4.12)$$

Remark 2 It should be pointed out that the schemes (7.4.6) and (7.4.10) are slightly different from (7.3.8) and (7.3.12). The techniques in the previous section result in only first order accuracy schemes, while schemes with arbitrary orders can be deduced by the methods in this section by choosing U_h as higher order element spaces.

Remark 3 As in §6.3, one may extend the methods in this section to systems of convection-diffusion equations by introducing a viscosity term (cf. [B-60]).

7.4.2 Convergence and error estimate

As in §7.3, we once again assume that $-\frac{1}{2}\text{div}b \geq \sigma_0 > 0$ and that T_h^* is a barycenter dual grid. By §3.2, there exists a constant $\alpha_0 > 0$ such that

$$a(\bar{u}_h, \Pi_h^* \bar{u}_h) \geq \alpha_0 \|\bar{u}_h\|_1^2, \quad \forall \bar{u}_h \in U_h. \quad (7.4.13)$$

Noting (6.2.15), we have a constant $\gamma_0 > 0$ such that

$$b_h(v_h, v_h) \geq \gamma_0 (\|v_h\|_0^2 + \|v_h\|_{\partial\Omega}^2). \quad (7.4.14)$$

First we need to evaluate the difference of $b(u, v_h)$ and $b_h(u, v_h)$. To this end, we observe that according to the approximation procedure of the convection term

$$\begin{aligned} b_h(u, v_h) = & - \sum_{j=1}^N \int_{K_{P_j}^*} \Pi_h^* u \text{div}(bv_h) dx \\ & + \sum_{j=1}^N \int_{\partial K_{P_j}^*} b \cdot \nu (\Pi_h^* u)^+ v_h ds, \end{aligned}$$

where $(\Pi_h^* u)^+$ stands for the upwind value across the boundary $\partial K_{P_j}^*$. Subtract it from (7.4.8a) to get

$$\begin{aligned} & b(u, v_h) - b_h(u, v_h) \\ = & - \sum_{j=1}^N \int_{K_{P_j}^*} (u - \Pi_h^* u) v_h(P_j) \text{div}b dx \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^N \left[\int_{(\partial K_{P_j}^*)_+} b \cdot \nu (u - u(P_j)) v_h(P_j) ds \right. \\
& \left. + \int_{(\partial K_{P_j}^*)_-} b \cdot \nu (u - (\Pi_h^* u)^+) v_h(P_j) ds \right] \\
& = I_{1h} + I_{2h},
\end{aligned} \tag{7.4.15}$$

where

$$\begin{aligned}
|I_{1h}| & = \left| \sum_{j=1}^N \int_{K_{P_j}^*} (u - \Pi_h^* u) v_h(P_j) \operatorname{div} b dx \right| \\
& = \left| \int_{\Omega} (u - \Pi_h^* u) v_h \operatorname{div} b dx \right| \\
& \leq C \|\operatorname{div} b\|_{\infty} h \|u\|_1 \|v_h\|_0,
\end{aligned} \tag{7.4.16a}$$

$$\begin{aligned}
I_{2h} & = \sum_{j=1}^N \sum_{l \in \Lambda_j} \int_{\Gamma_{jl}} b \cdot \nu [H(\beta_{jl})(u - u(P_j)) \\
& \quad + (1 - H(\beta_{jl}))(u - u(P_l))] v_h(P_j) ds.
\end{aligned} \tag{7.4.16b}$$

The above integral line Γ_{jl} 's are the sides of $K_{P_j}^*$. If $l \in \Lambda_j$, then along Γ_{lj} we have

$$\begin{aligned}
& \int_{\Gamma_{lj}} b \cdot \nu [H(\beta_{lj})(u - u(P_l)) \\
& \quad + (1 - H(\beta_{lj}))(u - u(P_j))] v_h(P_l) ds \\
& = - \int_{\Gamma_{jl}} b \cdot \nu [H(\beta_{jl})(u - u(P_j)) \\
& \quad + (1 - H(\beta_{jl}))(u - u(P_l))] v_h(P_j) ds.
\end{aligned}$$

So we have

$$\begin{aligned}
I_{2h} & = \frac{1}{2} \sum_{j=1}^N \sum_{l \in \Lambda_j} (v_h(P_j) - v_h(P_l)) \int_{\Gamma_{jl}} b \cdot \nu \\
& \quad [H(\beta_{jl})(u - u(P_j)) + (1 - H(\beta_{jl}))(u - u(P_l))] ds.
\end{aligned}$$

Similar to (7.2.27) one has

$$|I_{2h}| \leq C \|b\|_{\infty} h |u|_2 |\bar{u}_h|_1, \quad v_h = \Pi_h^* \bar{u}_h, \quad \bar{u}_h \in U_h. \quad (7.4.17)$$

Summarizing, we have a constant $C > 0$ such that

$$|b(u, \Pi_h^* \bar{u}_h) - b_h(u, \Pi_h^* \bar{u}_h)| \leq Ch \|u\|_2 \|\bar{u}\|_1. \quad (7.4.18)$$

After the above preparations, we readily obtain an estimate like (7.3.28) for the fully-discrete backward difference solution:

$$\max_{1 \leq k \leq N} \|u(t_k) - u_h^k\|_0 \leq Ch \|u\|_{X_1}. \quad (7.4.19)$$

Remark 4 If T_h is an acute triangulation, then the solution of the backward upwind scheme enjoys an extreme value property as well as a uniform convergence.

7.5 Upwind Schemes for Nonlinear Convection Problems

The above introduced generalized upwind schemes can be extended to elliptic and parabolic differential equations with a nonlinear convection term. The key point is to employ Osher's split technique of nonlinear functions. Retaining the previous symbols, we consider a nonlinear elliptic equation:

$$\begin{cases} -\mu \Delta u + \nabla \cdot F(x, u) = g(x, u), & x \in \Omega \subset \mathbb{R}^2, \\ u = 0, & x \in \Gamma = \partial\Omega, \end{cases} \quad (7.5.1a)$$

$$(7.5.1b)$$

where $F(x, u) = (f_1(x, u), f_2(x, u))$ is a smooth function on $\bar{\Omega} \times \mathbb{R}$, satisfying

$$F(x, 0) = 0. \quad (7.5.2)$$

Multiply (7.5.1) by v , integrate it on a dual element $K_{P_j}^*$, apply Green's formula, and sum it over $j = 1, 2, \dots, N$, then we have

$$a(u, v) + b(u, v) = (g, v), \quad (7.5.3)$$

where u and v satisfy (7.5.1b), and

$$a(u, v) = \sum_{j=1}^N \left[\mu \int_{K_{P_j}^*} \nabla u \cdot \nabla v dx - \mu \int_{\partial K_{P_j}^*} \frac{\partial u}{\partial \nu} v ds \right], \quad (7.5.4)$$

$$b(u, v) = - \sum_{j=1}^N \int_{K_{P_j}^*} F(x, u) \nabla v dx + \sum_{j=1}^N \int_{\partial K_{P_j}^*} F(x, u) \nu v ds. \quad (7.5.5)$$

Write F as

$$F(x, u) = \int_0^u \frac{\partial F(x, \bar{u})}{\partial \bar{u}} d\bar{u}. \quad (7.5.6)$$

Denote by P_{jl} the midpoint of two adjacent nodes P_j and P_l , and set

$$\beta_{jl}^+(u) = \int_0^u \max\left(0, \frac{\partial F(P_{jl}, \bar{u})}{\partial \bar{u}} \cdot \nu_{jl}\right) d\bar{u}, \quad (7.5.7)$$

$$\beta_{jl}^-(u) = \int_0^u \max\left(0, -\frac{\partial F(P_{jl}, \bar{u})}{\partial \bar{u}} \cdot \nu_{jl}\right) d\bar{u}, \quad (7.5.8)$$

where ν_{jl} is the unit outer normal direction of $\Gamma_{jl} \subset \partial K_{P_j}^*$. For $u_h \in U_h$ and $v_h \in V_h$, we introduce a bilinear form

$$b_h(u_h, v_h) = \sum_{j=1}^N v_h(P_j) \sum_{l \in \Lambda_j} \gamma_{jl} [\beta_{jl}^+(u_h(P_j)) - \beta_{jl}^-(u_h(P_l))], \quad (7.5.9)$$

where γ_{jl} is the length of Γ_{jl} ; U_h is the piecewise linear element space on T_h satisfying $U_h \subset H_0^1(\Omega)$; and V_h is the piecewise constant function space on T_h^* , subject to the zero boundary condition on $\partial\Omega$. Then, a generalized upwind difference scheme approximating (7.5.1) is: Find $u_h \in U_h$ such that

$$a(u_h, v_h) + b_h(u_h, v_h) = (g(x, u_h), v_h), \quad \forall v_h \in V_h. \quad (7.5.10)$$

This is apparently a generalization of Scheme (7.2.13). For a discussion of the monotonicity and convergence of (7.5.10), we refer to [B-85].

For a non-steady-state diffusion equation with a nonlinear convection term:

$$\frac{\partial u}{\partial t} - \mu \Delta u + \nabla \cdot F(x, u) = g(x, u), \quad (7.5.11)$$

we have the following upwind difference scheme:

$$(\bar{\partial}_t u_h^k, v_h) + a(u_h^{k,\theta}, v_h) + b_h(u_h^{k,\theta}, v_h) = (g(x, u_h^{k,\theta}), v_h), \quad (7.5.12)$$

where $u_h^{k,\theta} = \theta u_h^k + (1 - \theta)u_h^{k-1}$, $0 \leq \theta \leq 1$. (7.5.12) stands for a forward explicit scheme when $\theta = 0$, and a backward implicit scheme when $\theta = 1$.

In order to construct highly accurate upwind schemes, we rewrite (7.5.11) as

$$\frac{\partial u}{\partial t} - \mu \Delta u + \bar{b}(x, u) \cdot \nabla u = \bar{g}(x, u), \quad (7.5.13)$$

where

$$\bar{b}(x, u) = \left(\frac{\partial f_1}{\partial u}, \frac{\partial f_2}{\partial u} \right),$$

$$\bar{g}(x, u) = g(x, u) - \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right).$$

Define $a(u, v)$ by (7.5.4), and

$$b(u, v) = - \sum_{j=1}^N \int_{K_{P_j}^*} u \cdot \nabla (v \bar{b}) dx + \sum_{j=1}^N \int_{\partial K_{P_j}^*} (\bar{b} \cdot \nu) u v ds.$$

Then we can write (7.5.13) in a weak form: Find $u(x, t) \in C([0, T]; H_0^1(\Omega))$ such that

$$\left(\frac{\partial u}{\partial t}, v \right) + a(u, v) + b(u, v) = (\bar{g}, v), \quad \forall v \in H_0^1(\Omega). \quad (7.5.14)$$

In order to construct a upwind scheme, we first discretize the time direction to get

$$(\bar{\partial}_t u^{k-1}, v) + a(u^{k-1}, v) + b(u^{k-1}, v) = (\bar{g}(x, u^{k-1}), v).$$

Then set (cf (7.4.8b))

$$b_h(u_h^{k-1}, v_h) = \sum_{j=1}^N \int_{(\partial K_{P_j}^*)_-} \bar{b}(x, u_h^{k-1}) \cdot \nu [\Pi_h^* u_h^{k-1}] v_h ds. \quad (7.5.15)$$

So our task is to seek $u_h^k \in U_h$ such that

$$(\bar{\partial}_t u_h^k, v_h) + a(u_h^{k-1}, v_h) + b_h(u_h^{k-1}, v_h) = (\bar{g}(x, u_h^{k-1}), v_h),$$

$$\forall v_h \in V_h. \quad (7.5.16)$$

The stability and the convergence of schemes (7.5.12) and (7.5.16) have not been studied yet.

Bibliography and Comments

The paper [B-21] of Courant et. al. is the most fundamental work on the numerical solution of hyperbolic equations. [B26] combines characteristic methods with finite element, or finite difference, methods, and constructs a kind of upwind scheme on rectangular networks for the convection-dominated diffusion equations. Since 1977, Tabata and others have published a series of papers studying upwind schemes on triangular networks (cf. [B-3,83,84,85] and the references therein). They employ linear finite elements to discretize the diffusion term, and upwind difference schemes to discretize the convection term. Dong Liang ([A-17,18]) uses linear generalized difference method to deal with the diffusion term, and upwind schemes to convection term. Besides the methods discussed in the second section of this chapter, Dong Liang also proposes another class of upwind schemes based on some monotonic schemes, which is similar to some methods appearing in mechanics literature (cf. [B-80]). A class of highly accurate upwind schemes is obtained in [A-28] and [B-61] by approximating the diffusion term by higher order element generalized difference methods, and the convection term by highly accurate upwind schemes. We remark that the methods resulted from the linear case of this class of schemes are not identical to those in §7.2 and §7.3. [B-85] is among the few papers discussing nonlinear convection terms. Finally, we observe that if the diffusion coefficient $\mu = 0$, then the methods in this chapter result in the difference methods for hyperbolic equations.

Problem 1 Extend the results to a system of convection-dominated diffusion equations on higher dimensions, e.g., on two-dimensional regions.

Problem 2 Extend the results to higher dimensional, nonlinear, convection-dominated problems.

Chapter 8

APPLICATIONS

The first six sections of this chapter are devoted to the applications of generalized difference methods to elastic mechanics, fluid kinetics, electromagnetic fields, coupled sound-heat problems and long wave equations. The last section discusses the hierarchical basis methods for difference equations.

8.1 Planar Elastic Problems

Under certain conditions, one can regard the study of an elastic body, at an equilibrium state subject to an outer force, as a planar elastic problem. Let Ω be a planar region occupied by the elastic body, and $\Gamma = \partial\Omega$ its boundary. There are three groups of state variables: the stress tensor $\sigma = (\sigma_{11}, \sigma_{22}, \sigma_{12})^T$, the strain tensor $\epsilon = (\epsilon_{11}, \epsilon_{22}, \epsilon_{12})^T$, and the displacement tensor $u = (u_1, u_2)^T$. Assume the elastic body is homogeneous and isotropic. Write $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, denote by $\lambda, \mu > 0$ the Lamé constants, and set

$$E(\nabla) = \begin{pmatrix} \partial/\partial x_1 & 0 & \partial/\partial x_2 \\ 0 & \partial/\partial x_2 & \partial/\partial x_1 \end{pmatrix},$$
$$A = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix},$$

then σ, ϵ , and u satisfy the following three equations (cf. [A-26,16]):

$$\begin{cases} \epsilon = E^T(\nabla)u, & \text{(strain - displacement relation)} \end{cases} \quad (8.1.1)$$

$$\begin{cases} E(\nabla)\sigma + f = 0, & \text{(balance equation)} \end{cases} \quad (8.1.2)$$

$$\begin{cases} \sigma = A\epsilon, & \text{(stress - strain relation)} \end{cases} \quad (8.1.3)$$

where f is the body force.

It is an easy matter to deduce, from Green's formula, the following general Green's formula:

$$\int_{\Omega} \sigma^T E^T(\nabla)u dx + \int_{\Omega} (E(\nabla)\sigma)^T u dx = \int_{\partial\Omega} (E(\nu)\sigma)^T u ds, \quad (8.1.4)$$

where $\nu = (\nu_1, \nu_2)^T$ is the unit outer normal vector of Γ .

Suppose Γ is divided into two parts Γ_0 and Γ_1 . A displacement boundary condition $u = \bar{u}$ is given on Γ_0 , and a surface force condition $E(\nu)\sigma = \bar{P}$ on Γ_1 .

To solve the above system in practice, one usually eliminates some variables in (8.1.1)-(8.1.3), then solves it for the remaining unknowns, yielding accordingly the displacement method, the force method or the mixed method. In the sequel, we describe the generalized difference methods based on the displacement and the mixed methods.

8.1.1 Displacement methods

Eliminating σ and ϵ in (8.1.1)-(8.1.3) yields a system of second order elliptic partial differential equations of the displacement u :

$$-\mu \nabla^2 u - (\lambda + \mu) \text{grad div } u = f, \quad (8.1.5)$$

where $\text{div } u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}$. Multiply (8.1.5) by $v \in (H_E^1(\Omega))^2$, integrate it over $x \in \Omega$, and make use of Green's formula, then we have an equation in an integral form:

$$a(u, v) - \int_{\partial\Omega} \left(\mu \frac{\partial u}{\partial \nu} + (\lambda + \mu) (\text{div } u) \nu \right) v ds = (f, v), \quad (8.1.6)$$

where

$$a(u, v) = \int_{\Omega} [\mu \nabla u \nabla v + (\lambda + \mu) \text{div } u \cdot \text{div } v] dx. \quad (8.1.7)$$

On the boundary Γ_1

$$\mu \frac{\partial u}{\partial \nu} + (\lambda + \mu)(\operatorname{div} u)\nu = (E(\nu)\sigma)^T = \bar{P}.$$

Thus, we obtain a variational form of (8.1.5): Find $u \in (H^1(\Omega))^2$, $u|_{\Gamma_0} = u_0$ such that

$$a(u, v) = (f, v) + \int_{\Gamma_1} \bar{P} v ds, \quad \forall v \in (H_E^1(\Omega))^2. \quad (8.1.8)$$

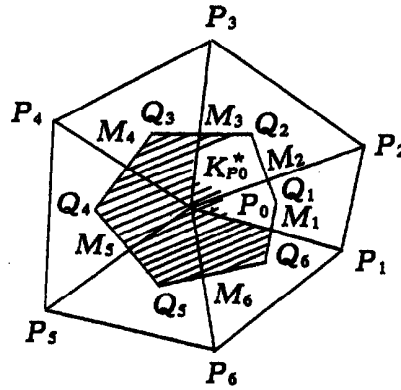


Fig. 8.1.1

To construct a generalized difference scheme, as before let $T_h = \{K\}$ be a triangulation of Ω such that $\Omega_h = \bigcup_{K \in T_h} K$ is an approximation of Ω . Let $T_h^* = \{K_P^*\}$ be a dual grid of T_h , usually a barycenter or a circumcenter dual grid. In the sequel, we assume T_h^* is a circumcenter dual grid. Fig. 8.1.1 shows all the triangular elements in T_h with vertex P_0 , as well as the dual element surrounding P_0 . Let U_h be a piecewise linear vector function space related to T_h . The interior nodes of T_h are numbered by $1, 2, \dots, N_0$. The boundary nodes are divided into two groups: The nodes where force conditions are given are numbered by $N_0 + 1, \dots, N_1$, and the nodes bearing displacement conditions by $N_1 + 1, \dots, N$. Denote by a scalar function $\phi_i(x)$ the basis function of the node $i \in \{1, 2, \dots, N_1\}$, then a function $u_h \in U_h$

satisfying $u|_{\Gamma_{0h}} = 0$ can be expressed as

$$u_h(x) = \sum_{i=1}^{N_1} u_i \phi_i(x),$$

where u_i is the value of $u_h(x)$ at the i -th node x_i , and Γ_{0h} is an approximation of Γ_0 . Choose V_h as the piecewise constant vector function space corresponding to T_h^* , subject to the boundary condition that $v_h \in V_h$ vanishes on the dual elements corresponding to the nodes $i \in \{N_1 + 1, \dots, N\}$. Let $\bar{\psi}_j = (\psi_j, \psi_j)^T$ be the dual basis function. The generalized difference equation reads: Find $u_h \in U_h$ such that $u_h = u_0$ on Γ_{0h} and

$$a(u_h, \bar{\psi}_j) = (f, \bar{\psi}_j) + \int_{\Gamma_{1h}} \bar{P} \bar{\psi}_j ds, \quad j = 1, 2, \dots, N_1, \quad (8.1.9)$$

where Γ_{1h} is a certain approximation of Γ_1 .

In fact, we can as well work out (8.1.9) in a more direct manner. So we integrate the two sides of (8.1.5), apply Green's formula, and replace u by u_h to obtain

$$-\int_{\partial K_{P_0}^*} \left[\mu \frac{\partial u_h}{\partial \nu} + (\lambda + \mu)(\operatorname{div} u_h) \nu \right] ds = \int_{K_{P_0}^*} f dx. \quad (8.1.10)$$

This is the generalized difference equation at the node P_0 . Let P_0 and its adjacent nodes be as in Fig. 8.1.1. We now compute the integrals on the left-hand side of (8.1.10). We divide the first integral into a sum of integrals on the perpendicular bisector segments $\overline{Q_1 Q_2}, \overline{Q_2 Q_3}, \dots, \overline{Q_6 Q_1}$. For instance, the integral on $\overline{Q_1 Q_2}$ looks like

$$-\mu \int_{\overline{Q_1 Q_2}} \frac{\partial u_h}{\partial \nu} ds = -\mu |\overline{Q_1 Q_2}| (u_{P_2} - u_{P_0}) / |\overline{P_0 P_2}|. \quad (8.1.11)$$

Similarly, the second integral of (8.1.10) is divided into a sum of integrals on the fold line segments $\overline{M_1 Q_1 M_2}, \overline{M_2 Q_2 M_3}, \dots, \overline{M_6 Q_6 M_1}$. For example, on $\overline{M_1 Q_1 M_2}$,

$$\begin{aligned} & -(\lambda + \mu) \int_{\overline{M_1 Q_1 M_2}} (\operatorname{div} u_h) \nu ds \\ &= -(\lambda + \mu) \operatorname{div} u_h(Q_1) \left(\frac{\overline{P_0 P_1}}{|\overline{P_0 P_1}|} |\overline{M_1 Q_1}| + \frac{\overline{P_0 P_2}}{|\overline{P_0 P_2}|} |\overline{Q_1 M_2}| \right), \end{aligned} \quad (8.1.12)$$

where

$$\begin{aligned}
 \operatorname{div} u_h(Q_1) &= \frac{\partial u_{1h}}{\partial x_1}(Q_1) + \frac{\partial u_{2h}}{\partial x_2}(Q_1) \\
 &= \frac{1}{2S_{Q_1}} [(x_2(P_1) - x_2(P_2))u_1(P_0) + (x_2(P_2) - x_2(P_0))u_1(P_1) \\
 &\quad + (x_2(P_0) - x_2(P_1))u_1(P_2)] + \frac{1}{2S_Q} [(x_1(P_2) - x_1(P_1))u_2(P_0) \\
 &\quad + (x_1(P_0) - x_1(P_2))u_2(P_1) + (x_1(P_1) - x_1(P_0))u_2(P_2)].
 \end{aligned}
 \tag{8.1.13}$$

Here $(x_1(P_i), x_2(P_i))$ are the coordinates of the node P_i , and S_{Q_1} is the area of the triangular element containing the circumcenter Q_1 .

Equation (8.1.10) is a generalized difference equation for an interior node. On the boundary Γ_{0h} the displacement u_0 is given. Extra equations are needed for the nodes on Γ_{1h} . As an example, suppose a node $\bar{P}_0 \in \Gamma_{1h}$ and its neighbouring nodes are as in Fig. 8.1.2. In such a case, we still have equation (8.1.10), and the line integrals on the left-hand side are computed by use of the formulas (8.1.11) and (8.1.12).

So we finally end up with a system of $(2N_1)$ equations like (8.1.10) together with the displacement boundary condition on Γ_{0h} .

Compared with the (piecewise linear) finite element method, our generalized difference method here enjoys the same convergence order, and less computational work. One can also employ high order generalized difference methods to approximate planar elastic problems. In practical computation, it might be more convenient to use a barycenter dual grid instead, since it is more accurate and can be extended directly to three-, or arbitrary n -, dimensional problems.

8.1.2 Mixed methods

Work out $\epsilon = B\sigma$ ($B = A^{-1}$) from (8.1.3), insert it into (8.1.1), and couple it with (8.1.2), then we have the following system of equations of σ and u :

$$B\sigma - E^T(\nabla)u = 0, \tag{8.1.14}$$

$$E(\nabla)\sigma = f, \tag{8.1.15}$$

where we have replaced f by $-f$, and

$$B = \frac{1}{4\mu(\lambda + \mu)} \begin{bmatrix} \lambda + 2\mu & -\lambda & 0 \\ -\lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 4(\lambda + \mu) \end{bmatrix}. \quad (8.1.16)$$

In this case the displacement condition remains to be an essential boundary condition, while the force condition is a natural boundary condition. The space U_h of the approximate displacement and the corresponding test function space V_h are constructed as before, with nodal basis functions $\phi_i(x)$ and $\bar{\psi}_i(x)$ respectively.

The three-dimensional tensor σ_k 's belong to a piecewise linear (vector) function space M_h related to T_h , of which the vertexes of T_h are the interpolation nodes and the nodal basis function is $\xi_i(x)$. The corresponding test function space N_h is the piecewise constant space with respect to T_h^* , with nodal basis function $\bar{\eta}_i(x)$. Define (cf. [A-26])

$$a(\sigma \times u, \tau \times v) = \int_{\Omega} \{ \tau^T B \sigma - \tau^T E^T u - \sigma^T E^T (\nabla) v \} dx.$$

Then, the generalized difference method for the equations (8.1.14) and (8.1.15) is: Find $u_h \in U_h$, $u_h|_{\Gamma_{0h}} = u_0$ and $\sigma_h \in M_h$ such that

$$a(\sigma_h \times u_h, \bar{\eta}_j \times \bar{\psi}_j) = (f, \bar{\psi}_j) + \int_{\Gamma_{1h}} \bar{P} \bar{\psi}_j ds, \quad j = 1, 2, \dots, N_1. \quad (8.1.17)$$

Once again, we have a more direct way to deduce the generalized difference equation. Integrate (8.1.14) and (8.1.15) respectively on a dual element $K_{P_0}^*$ and apply the following Green's formulas:

$$\begin{aligned} \int_{K_{P_0}^*} E^T(\nabla) u_h dx &= \int_{\partial K_{P_0}^*} (E(\nu))^T u_h ds, \\ \int_{K_{P_0}^*} (E(\nabla) \sigma_h)^T dx &= \int_{\partial K_{P_0}^*} (E(\nu) \sigma_h)^T ds, \end{aligned}$$

then we have

$$\int_{K_{P_0}^*} B \sigma_h dx - \int_{\partial K_{P_0}^*} (E(\nu))^T u_h ds = 0, \quad (8.1.14)'$$

$$\int_{\partial K_{P_0}^*} (E(\nu)\sigma_h)^T ds = \int_{K_{P_0}^*} f dx. \quad (8.1.15)'$$

Suppose P_0 and its neighbouring nodes are as in Fig. 8.1.1. Then the double integral on $K_{P_0}^*$ can be divided into a sum of integrals on the intersections of $K_{P_0}^*$ and the adjacent elements respectively. For example, we have

$$\begin{aligned} & \int_{\Delta P_0 P_1 P_2 \cap K_{P_0}^*} B \sigma_h dx \\ &= \frac{1}{3} B \{ (S_{P_0 M_1 Q_1} + S_{P_0 Q_1 M_2}) (\sigma_h(P_0) + \sigma_h(Q_1)) \\ & \quad + S_{P_0 M_1 Q_1} \sigma_h(M_1) + S_{P_0 Q_1 M_2} \sigma_h(M_2) \}, \end{aligned}$$

where S_{ABC} denotes the area of a triangle $\triangle ABC$. Similarly, the line integral on $\partial K_{P_0}^*$ can be divided into a sum of the line integrals on the perpendicular bisectors. For instance,

$$\begin{aligned} & \int_{\overline{Q_1 Q_2}} E(\nu)^T u_h ds \\ &= (E(\nu))^T (|\overline{Q_1 M_2}| u_h(Q_1) + |\overline{Q_1 Q_2}| u_h(M_2) + |\overline{M_2 Q_2}| u_h(Q_2)) / 2, \\ & \quad \nu = \overline{P_0 P_2} / |\overline{P_0 P_2}|. \end{aligned}$$

So we end up with a system of $(5N_1)$ generalized difference equations of the forms (8.1.14)' and (8.1.15)', plus the displacement boundary conditions on Γ_{0h} . Like the finite element method, the generalized difference method based on the mixed variational form can obtain both the displacement and the strain simultaneously.

8.2 Computation of Electromagnetic Fields

In 1967, A. M. Winslow [B-99] applied a difference method on irregular networks to a two-dimensional quasi-linear Poisson equation representing an electromagnetic field. He allows the network to be a planar curvilinear network, but each node can be the vertex of at most six triangular elements. In 1990, G. Zhao and Y. Liu [A-60] extended Winslow's method to three dimensions, and carried out a

numerical experiment for a tetrapolar lens. Their numerical result matches well the theoretical prediction. They adopt cylindrical coordinates. Then they cut the region, for different angles θ , to get some (r, z) planes D_i 's, on which is placed a Winslow triangulation with curvilinear sides. Special cones are constructed between different D_i 's, referred to as a "secondary network" (corresponding to the dual grid). This paper claims that this method has a bright future in the computation of photoelectronics. We describe in this section in a united framework the generalized difference method for three-dimensional Poisson equations. We only present the results for the Poisson equation in Cartesian coordinates, which can be directly extended to cylindrical or spheroidal coordinates, and even to the second order elliptic equations with variable coefficients. As in the planar case, one can similarly establish the convergence and the error estimate, which are omitted here.

Let $\Omega \subset \mathbb{R}^3$ be a polyhedral region, and $T_h = \{K\}$ a tetrahedral grid of Ω such that different tetrahedral elements share no common interior and $\Omega = \bigcup_{K \in T_h} K$. As in the case of planar triangular elements, we can analogously introduce a tetrahedral volume coordinates. Let the vertexes of a tetrahedron K be P_i ($i = 1, 2, 3, 4$). Then for any point $P \in K$ the volume coordinates $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ are defined as

$$\lambda_i = \frac{V_i}{V}, \quad i = 1, 2, 3, 4,$$

where V is the volume of K , and V_i is the volume of the tetrahedron formed by the point P and the base triangle facing P_i (cf. Fig. 8.2.1). Apparently $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$.

If the Cartesian coordinates of P_i ($i = 1, 2, 3, 4$) are (x_i, y_i, z_i) , and the volume coordinates of P are $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, then the Cartesian coordinates (x, y, z) of P can be expressed by the volume coordinates as

$$\begin{cases} x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4, \\ y = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \lambda_4 y_4, \\ z = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 + \lambda_4 z_4, \\ 1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4. \end{cases} \quad (8.2.1)$$

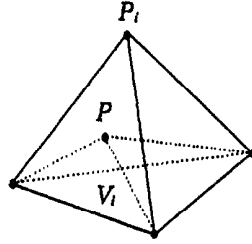


Fig. 8.2.1

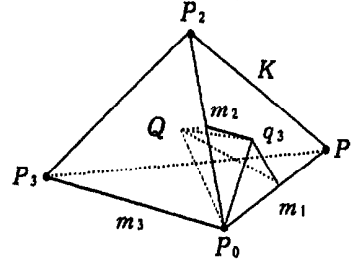


Fig. 8.2.2

On the other hand, the volume coordinates can be expressed by the Cartesian coordinates as

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \frac{1}{6V} \begin{bmatrix} X_{14} & Y_{14} & Z_{14} \\ X_{24} & Y_{24} & Z_{24} \\ X_{34} & Y_{34} & Z_{34} \end{bmatrix} \begin{bmatrix} x - x_4 \\ y - y_4 \\ z - z_4 \end{bmatrix}, \quad (8.2.2)$$

where

$$X_{i4} = \begin{vmatrix} y_{j4} & y_{k4} \\ z_{j4} & z_{k4} \end{vmatrix}, \quad Y_{i4} = \begin{vmatrix} z_{j4} & z_{k4} \\ x_{j4} & x_{k4} \end{vmatrix}, \quad Z_{i4} = \begin{vmatrix} x_{j4} & x_{k4} \\ y_{j4} & y_{k4} \end{vmatrix},$$

$$x_{j4} = x_j - x_4, \quad y_{j4} = y_j - y_4, \quad z_{j4} = z_j - z_4.$$

$$(j = i + 1, \quad k = j + 1, \quad i = k + 1.)$$

$V = |\bar{V}|$ is the area of the element, given by

$$\bar{V} = \frac{1}{6} \begin{vmatrix} x_{14} & x_{24} & x_{34} \\ y_{14} & y_{24} & y_{34} \\ z_{14} & z_{24} & z_{34} \end{vmatrix} = -\frac{1}{6} \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}. \quad (8.2.3)$$

Now we define the dual grid T_h^* . Let P_0 be a node (cf. Fig. 8.2.2). Consider all the tetrahedrons with P_0 as a vertex. K is one of them, depicted in Fig. 8.2.2, with vertexes P_0, P_1, P_2, P_3 . K has three base triangles with the vertex P_0 , one of which is $\triangle P_0 P_1 P_2$. Denote

the midpoints of $\overline{P_0P_1}$ and $\overline{P_0P_2}$ by m_1 and m_2 respectively, and the barycenter of $\triangle P_0P_1P_2$ by q_3 . Q represents the barycenter of K , with volume coordinates $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Take P_0 as a vertex, and $\triangle Qq_3m_1$ and $\triangle Qq_3m_2$ as bases respectively to form two cones. For each of the other two triangles $\triangle P_0P_2P_3$ and $\triangle P_0P_1P_3$, we similarly define two cones. The union of all these six cones constitutes the intersection of the element K and the dual element $K_{P_0}^*$ surrounding P_0 . By taking P_0 to be every (inner and boundary) node, we end up with a (barycenter) dual grid T_h^* related to T_h .

The trial function space U_h corresponding to the grid T_h is chosen as the usual finite element space such that $u_h \in U_h$ is a piecewise polynomial of degree k on $T_h = \{K\}$, possessing a global smoothness of a certain degree. Here we restrict ourselves to discuss only the piecewise linear function space with the vertexes of the tetrahedrons as the nodes. On a tetrahedron with vertexes P_0, P_1, P_2, P_3 , the trial function is of the form:

$$\phi(x, y, z) = u_0\lambda_0 + u_1\lambda_1 + u_2\lambda_2 + u_3\lambda_3, \quad (8.2.4)$$

where $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ is the volume coordinates of (x, y, z) . Obviously we have that $\phi(P_i) = u_i$ for $i = 0, 1, 2, 3$, that any $u_h \in U_h$ is globally continuous, and that $U_h \subset H^1(\Omega)$. The test function space V_h related to T_h^* is taken as a piecewise polynomial space of degree k . V_h is not required to have any global smoothness, but it has the same dimension as U_h . Each node P_0 bears some nodal basis functions, of which the number depends on the type and the number of the interpolation conditions of U_h at P_0 . For instance, if U_h is a piecewise linear function space, then the nodal basis function of V_h at P_0 is

$$\psi_{P_0}(x, y, z) = \begin{cases} 1, & \text{for } (x, y, z) \in K_{P_0}^*, \\ 0, & \text{elsewhere.} \end{cases}$$

If U_h is a standard finite element space, then the basis functions of V_h are of the form

$$\psi_{P_0}(x, y, z) = \begin{cases} \frac{1}{l!m!n!}(x-x_0)^l(y-y_0)^m(z-z_0)^n, & (x, y, z) \in K_{P_0}^*, \\ 0, & \text{elsewhere.} \end{cases}$$

Next we consider the generalized difference method for the following Poisson equation

$$\begin{cases} \nabla(a\nabla u) = \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(a\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(a\frac{\partial u}{\partial z}\right) = f, & \text{on } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \quad (8.2.5)$$

where $a = a(x, y, z) \geq a_0 > 0$. Assume U_h and V_h are piecewise linear and piecewise constant function spaces, respectively. Integrate (8.2.5) on $K_{P_0}^*$ and use the Gauss formula to obtain

$$\int_{\partial K_{P_0}^*} a \frac{\partial u}{\partial \nu} ds = \int_{K_{P_0}^*} f dx dy dz, \quad (8.2.6)$$

where ν is the unit outer normal direction of $\partial K_{P_0}^*$. We see from Fig. 8.2.2 that $\partial K_{P_0}^*$ is cut into six planar triangles with a common vertex (the barycenter) Q , by the tetrahedron K with vertexes P_0, P_1, P_2, P_3 . These six triangles are divided into three pairs, with the barycenters of $\triangle P_0 P_1 P_2$, $\triangle P_0 P_2 P_3$, and $\triangle P_0 P_1 P_3$ respectively as a common vertex. Now we can deduce the surface integral in (8.2.6) into a sum of integrals on these triangles. Notice

$$\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu, \quad (8.2.7)$$

where ν is the unit outer normal directions of these triangles on $\partial K_{P_0}^*$. Also note that ∇u is a constant vector on K . Thus we may use (8.2.2) and (8.2.4) (changing the numbering 1,2,3,4 into 0,1,2,3) to get

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right), \quad (8.2.8)$$

$$\begin{aligned} \frac{\partial u}{\partial x} &= u_0 \frac{\partial \lambda_0}{\partial x} + u_1 \frac{\partial \lambda_1}{\partial x} + u_2 \frac{\partial \lambda_2}{\partial x} + u_3 \frac{\partial \lambda_3}{\partial x}, \\ \frac{\partial u}{\partial y} &= u_0 \frac{\partial \lambda_0}{\partial y} + u_1 \frac{\partial \lambda_1}{\partial y} + u_2 \frac{\partial \lambda_2}{\partial y} + u_3 \frac{\partial \lambda_3}{\partial y}, \\ \frac{\partial u}{\partial z} &= u_0 \frac{\partial \lambda_0}{\partial z} + u_1 \frac{\partial \lambda_1}{\partial z} + u_2 \frac{\partial \lambda_2}{\partial z} + u_3 \frac{\partial \lambda_3}{\partial z}, \end{aligned} \quad (8.2.9)$$

where

$$\begin{cases} \frac{\partial \lambda_i}{\partial x} = \frac{1}{6V} X_{i4}, \quad i = 0, 1, 2, \\ \frac{\partial \lambda_3}{\partial x} = -\frac{1}{6V} (X_{04} + X_{14} + X_{24}), \\ X_{i4} = \begin{vmatrix} y_{j4} & y_{k4} \\ z_{j4} & z_{k4} \end{vmatrix}, \end{cases} \quad (8.2.10)$$

where the subscripts $j = i + 1$, $k = j + 1$, $i = k + 1$. The representations of $\partial \lambda_i / \partial y$ and $\partial \lambda_i / \partial z$ ($i = 0, 1, 2, 3$) can be obtained in like manner. The normal vector ν depends on the different triangles on $\partial K_{P_0}^*$. For example, the outer normal direction of ΔQq_3m_1 and ΔQq_3m_2 are respectively

$$\frac{\overline{q_3Q} \times \overline{q_3m_1}}{|\overline{q_3Q}| |\overline{q_3m_1}|}, \quad \frac{\overline{q_3m_2} \times \overline{q_3Q}}{|\overline{q_3m_2}| |\overline{q_3Q}|}.$$

Here the symbol $\times$ denotes the vector product. As regards the coefficient a , we usually take its average value on the vertexes of ΔQq_3m_i :

$$a_{3i} = \frac{1}{3}(a(Q) + a(q_3) + a(m_i)), \quad i = 1, 2.$$

Summarizing, we can deduce (8.2.5) into the following difference equation:

$$\sum_K (\nabla u)_K \sum a_{3i} \text{meas}(\Delta Qq_3m_i) \nu_{\Delta Qq_3m_i} = \int_{K_{P_0}^*} f dx dy dz. \quad (8.2.11)$$

In practice, in the interior of Ω , we often first place a cuboid grid, and then divide each cuboid element into six tetrahedrons, while on the boundary of Ω we use directly a tetrahedral grid. So we get a difference equation on a rectangular networks.

In [A-60] the equation (8.2.5) is written in cylindrical coordinates:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(ar \frac{\partial u}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{a}{r} \frac{\partial u}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(a \frac{\partial u}{\partial z} \right) = -p.$$

When a is a constant, this equation is simplified as

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{p}{a}.$$

In [A-60], first they place on (r, z) plane a barycenter dual grid, then they connect certain nodes in between the surfaces $\theta = \theta_l, \theta_m, \theta_n$ into some polyhedrons to finally form a dual element. There it is allowed for different dual elements to have overlapping interiors. Thus their method differs from ours here.

8.3 Numerical Simulation of Underground Water Pollution

Underground water is often contaminated by, e.g., the sewage out of factories or mines, and the chemical fertilizer and pesticide in agriculture, which seep into the ground with rain or irrigation and drainage. These solutes in the water may perform a convective motion with respect to the underground water, and/or a diffusive motion due to the density diffusion of the water molecules. In hydrogeology and environmental science, computer applications using mathematical models are widely used to study the law of the motion of the contaminated water. A mathematical model describing the contaminated water, or the water with any chemical solute (e.g., saline-alkali) in general, is the following equation of the solute density C (cf. [A-40]):

$$\frac{\partial(mC)}{\partial t} = \operatorname{div}(mD\operatorname{grad}C) - \operatorname{div}(VmC) - \frac{C'W}{n}, \text{ on } \Omega \subset \mathbb{R}^l. \quad (8.3.1)$$

Here $l = 1, 2$ or 3 . To fix the idea, we take $l = 2$. The other notations are explained below.

m : The saturation thickness of the water-bearing formation, usually depending on (x, y, z) .

$V = (V_x, V_y)^T$: The velocity of the water, assumed to be known.

$D = \begin{pmatrix} D_{xx} & D_{xy} \\ D_{xy} & D_{yy} \end{pmatrix}$: The diffusion coefficient tensor, depending on the composition of the solute.

W : The amount of the water flooded into (positive) or pumped off from (negative) a unit area of water-bearing

ing formation. In particular, if the water goes in or out through a well $P_0 = (x_0, y_0)$, i.e., P_0 is either a source or a sink, then $W = Q\delta(x - x_0, y - y_0)$, where Q is the amount of the water.

C' : The density of the solute, which is known for a source, and unknown for a sink.

Besides, the boundary and initial conditions are also needed to be provided.

The first term appearing in the right-hand side of (8.3.1) is referred to as a diffusion term, while the second a convection term, and the third a source term. In fact, the differential equation (8.3.1) may describe many other physical or chemical phenomena, so long as C and the other notations are explained accordingly. For instance, C may denote the mass ratio of chemical composition, the heat enthalpy, the temperature, or the kinetic energy of turbulent flow, etc. (cf. [B-73].)

8.3.1 Generalized difference scheme

Let us place a triangulation $T_h = \{K\}$ and its dual grid $T_h^* = \{K^*\}$ (barycenter or circumcenter dual grids, cf. Fig. 8.3.1). Both the sources and the sinks must be taken as nodes. If the coefficient of the diffusion term is discontinuous on a line L , then L should be cut into several line segments by some nodes, such that each segment is a side of an element. We assume that C is continuous crossing such an L , i.e., $C_+ = C_-$, where $+$ and $-$ denote the two sides of L respectively. The flow of the solute is also assumed to be continuous:

$$(m(D\text{grad}C) \cdot \nu)_+ = (m(D\text{grad}C) \cdot \nu)_-,$$

where ν is the unit outer normal vector of L . Under these assumptions, a generalized differencing of (8.3.1) at a node on L can be done precisely as at other nodes.

Let T_h^* be a barycenter dual grid. A node P_0 together with its neighbouring nodes and corresponding dual elements are depicted in Fig. 8.3.1. The trial function space U_h is the piecewise linear function

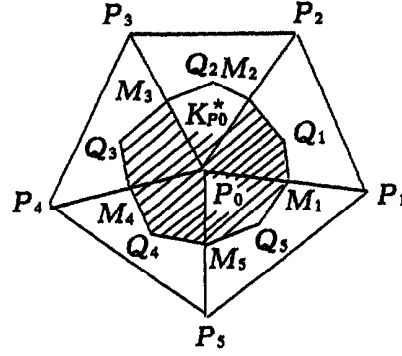


Fig. 8.3.1

space related to T_h with the vertexes of the elements as the nodes, and the test function space is the piecewise constant space corresponding to T_h^* . Integrate (8.3.1) on $K_{P_0}^*$ to obtain

$$\begin{aligned} & \int_{K_{P_0}^*} \frac{\partial(mC)}{\partial t} dx dy \\ &= \int_{K_{P_0}^*} \operatorname{div}(mD\operatorname{grad}C) dx dy - \int_{K_{P_0}^*} \operatorname{div}(VmC) dx dy \quad (8.3.2) \\ & \quad - \int_{K_{P_0}^*} \frac{C'W}{n} dx dy. \end{aligned}$$

By Green's formula we have for $C = C_h \in U_h$

$$\int_{K_{P_0}^*} \operatorname{div}(mD\operatorname{grad}C_h) dx dy = \int_{\partial K_{P_0}^*} m(D\operatorname{grad}C_h) \cdot \nu ds, \quad (8.3.3)$$

$$\int_{K_{P_0}^*} \operatorname{div}(VmC_h) dx dy = \int_{\partial K_{P_0}^*} mC_h(V \cdot \nu) ds. \quad (8.3.4)$$

Denote by Δ_{Q_i} a triangular element with a barycenter Q_i . Evaluate the above line integral piecewise on the fold line segments obtained by intersecting the integral line with Δ_{Q_i} . One can readily work out the formula for the piecewise integrals thanks to the linearity of $u_h \in U_h$.

For example, in Δ_{Q_1} we have

$$\begin{aligned} & \int_{\overline{M_1 Q_1 M_2}} m(D\text{grad} C_h) \cdot \nu ds \\ &= \int_{\overline{M_1 Q_1}} m\text{grad} C_h \cdot D\nu_1 ds + \int_{\overline{Q_1 M_2}} m\text{grad} C_h \cdot D\nu_2 ds. \end{aligned} \quad (8.3.3a)$$

where

$$\begin{aligned} \text{grad} C_h &= \frac{1}{2\Delta_{Q_1}} ((y_{P_1} - y_{P_2})C_0 + (y_{P_2} - y_{P_0})C_1 \\ &\quad + (y_{P_0} - y_{P_1})C_2 + (x_{P_2} - x_{P_1})C_0 \\ &\quad + (x_{P_0} - x_{P_2})C_1 + (x_{P_1} - x_{P_0})C_2), \end{aligned} \quad (8.3.3b)$$

$$\begin{cases} \nu_1 = (y_{Q_1} - y_{M_1}, -(x_{Q_1} - x_{M_1}))/|\overline{M_1 Q_1}|, \\ \nu_2 = (y_{M_2} - y_{Q_1}, -(x_{M_2} - x_{Q_1}))/|\overline{Q_1 M_2}|, \end{cases} \quad (8.3.3c)$$

where (x_P, y_P) denote the coordinates of a point P . Also note

$$\begin{aligned} & \int_{\overline{M_1 Q_1 M_2}} mC_h(V \cdot \nu) ds \\ &= \int_{\overline{M_1 Q_1}} mC_h(V \cdot \nu_1) ds + \int_{\overline{Q_1 M_2}} mC_h(V \cdot \nu_2) ds, \end{aligned} \quad (8.3.4a)$$

where ν_1 and ν_2 are as in (8.3.3c). On the line segment $\overline{M_1 Q_1}$: $y = y_{M_1} + \frac{y_{Q_1} - y_{M_1}}{x_{Q_1} - x_{M_1}}(x - x_{M_1})$, we have

$$C_h = \frac{x_{Q_1} - x}{x_{Q_1} - x_{M_1}} C_{M_1} + \frac{y - y_{M_1}}{y_{Q_1} - y_{M_1}} C_{Q_1}. \quad (8.3.4b)$$

Similarly on $\overline{Q_1 M_2}$: $y = y_{Q_1} + \frac{y_{M_2} - y_{Q_1}}{x_{M_2} - x_{Q_1}}(x - x_{Q_1})$, we have

$$C_h = \frac{x_{M_2} - x}{x_{M_2} - x_{M_1}} C_{Q_1} + \frac{y - y_{Q_1}}{y_{M_2} - y_{Q_1}} C_{M_2}. \quad (8.3.4c)$$

Also observe that

$$C_{Q_1} = \frac{1}{3}(C_{P_0} + C_{P_1} + C_{P_2}), \quad C_{M_i} = \frac{1}{2}(C_{P_0} + C_{P_i}). \quad (8.3.5)$$

As regards the source term on the right-hand side of (8.3.2), if P_0 is not a well, then it is directly computed (C' is known for a source, and unknown for a sink). On the other hand, if P_0 is a well, then

$$W = Q\delta(x - x_{P_0}, y - y_{P_0}).$$

In this case

$$\int_{K_{P_0}^*} \frac{C'Q}{n} \delta(x - x_{P_0}, y - y_{P_0}) dx dy = \frac{C'(P_0)}{n(P_0)} Q(P_0). \quad (8.3.6)$$

Now we have successfully discretized the space variable on the right-hand side of (8.3.2). To further discretize the derivative with respect to t on the left-hand side, we take a time step size $\tau > 0$ and nodes $t_k = k\tau$ ($k = 0, 1, \dots, K$). We can use explicit or implicit Euler's methods, or Crank-Nicolson method to approximate (8.3.2). For example, the Crank-Nicolson method gives

$$\begin{aligned} & \tau^{-1} \int_{K_{P_0}^*} (m^{k+1} C_h^{k+1} - m^k C_h^k) dx dy \\ &= \frac{1}{2} \int_{K_{P_0}^*} \operatorname{div}(m^{k+1/2} D \operatorname{grad}(C_h^k + C_h^{k+1})) dx dy \\ & \quad - \frac{1}{2} \int_{K_{P_0}^*} \operatorname{div}(V^{k+1/2} m^{k+1/2} (C_h^k + C_h^{k+1})) dx dy \\ & \quad - \frac{1}{2} \int_{K_{P_0}^*} \frac{W^k}{n} C_h^k dx dy. \end{aligned} \quad (8.3.7)$$

This together with boundary and initial conditions gives the generalized difference scheme for (8.3.1). Quite a few applications of generalized difference methods are discussed in [A-40].

Remark If a circumcenter grid T_h^* is adopted, then the related computation will be simpler, resulting in a difference method on irregular networks, as is called in the underground water computation. As in §8.2, we may extend the generalized difference scheme to tetrahedral, cuboid or triangular prismatic grids on a three-dimensional field. (cf. [A-40].)

8.3.2 Generalized upwind difference schemes

Generalized difference schemes have been widely and, generally speaking, satisfactorily used in the underground water computation. But in the computation of contaminated underground water, one often encounters a class of problems where the diffusion coefficient is much less than the convection speed. In such a case, a standard generalized difference method fails to approximate accurately the transitional band that results from the diffusion, and it may bring in undesirable oscillations. Upwind difference schemes are often used to overcome this difficulty. To illustrate the idea, let us investigate a one-dimensional convection-diffusion equation ([A-40]):

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} - V \frac{\partial C}{\partial x}, \quad (8.3.8a)$$

where D and V are positive constants. The initial and boundary conditions are

$$\begin{cases} C(x, 0) = 0, & x > 0, \\ C(0, t) = C_0, & t \geq 0, \\ C(\infty, t) = 0, & t \geq 0. \end{cases} \quad (8.3.8b)$$

The true solution to this problem is

$$C(x, t) = \frac{C_0}{2} \left\{ \operatorname{erfc} \left(\frac{x - Vt}{2\sqrt{Dt}} \right) + \exp \left(\frac{Vx}{D} \right) \operatorname{erfc} \left(\frac{x + Vt}{2\sqrt{Dt}} \right) \right\}.$$

$$\left(\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt. \right)$$

Apply the generalized difference method to (8.3.8a) to obtain the following implicit difference scheme:

$$\frac{C_j^{k+1} - C_j^k}{\tau} = D \frac{C_{j+1}^{k+1} - 2C_j^{k+1} + C_{j-1}^{k+1}}{h^2} - V \frac{C_{j+1}^{k+1} - C_{j-1}^{k+1}}{2h}.$$

Define the Peclet number as

$$Pe = Vh/D.$$

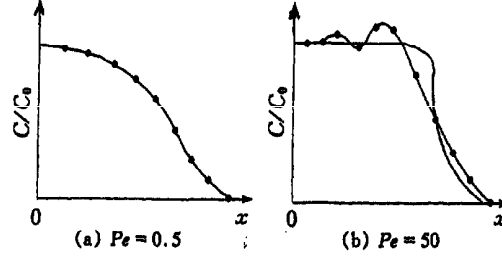


Fig. 8.3.2

For fixed step size $h > 0$, Pe varies with the ratio V/D . A comparison of the true solution and the implicit difference solution at $t = 50\tau$ is depicted in Fig. 8.3.2(a) for $Pe = 0.5$ and (b) for $Pe = 50$. We observe that the approximation is fairly good for a small Peclet number. But for a large Peclet number, i.e., for the convection-dominated diffusion case, the density front becomes wider and more level, and undesirable oscillations appear. The too wide and level band is caused by an extra diffusion in the numerical discretization, which can be decreased by a smaller step size. On the other hand, the oscillation is due to the approximation of the convection term by a central differencing. Upwind schemes are often used to eliminate the oscillation as discussed below.

Consider a two-dimensional solute transfer equation

$$\frac{\partial C}{\partial t} = \text{div}(D \text{grad} C) - \text{div}(VC) + I, \quad (8.3.9)$$

where the diffusion tensor D and the convection speed V are known, as in equation (8.3.2), and I is the source term with $I = C'Q\delta(x - x_0, y - y_0)$ at a well. As before, assume $T_h = \{K\}$ is a triangulation, and $T_h^* = \{K_{P_0}^*\}$ is a barycenter dual grid (cf. Fig. 8.3.1). Let U_h be the piecewise linear, globally continuous function space, and V_h the piecewise constant function space. Denote by Π_h^* the interpolation projection operator from U_h to V_h : For given $C_h \in U_h$, $\Pi_h^* C_h \in V_h$ and $(\Pi_h^* C_h)(P_0) = C_h(P_0)$. Integrate equation (8.3.9) on $K_{P_0}^*$, set $C = C_h \in U_h$, and replace C_h in the convection term by $\Pi_h^* C_h$, then

we have

$$\begin{aligned} \int_{K_{P_0}^*} \frac{\partial C_h}{\partial t} dx dy &= \int_{K_{P_0}^*} \operatorname{div}(D \operatorname{grad} C_h) dx dy \\ &- \int_{K_{P_0}^*} \operatorname{div}(V \Pi_h^* C_h) dx dy + \int_{K_{P_0}^*} I dx dy. \end{aligned} \quad (8.3.10)$$

The diffusion term on the right-hand side is computed according to (8.3.3) and (8.3.3a-c) with $m = 1$, while the source term according to (8.3.6) with $n = 1$. The convection term is treated as in §7.2. To elaborate, we apply Green's formula

$$\int_{K_{P_0}^*} \operatorname{div}(V \Pi_h^* C_h) dx dy = \int_{\partial K_{P_0}^*} (V \cdot \nu) \Pi_h^* C_h ds. \quad (8.3.11)$$

Set (cf. Fig. 8.3.1) $\Gamma_{0l\delta} = \overline{Q_l M_{l+\delta}}$, $l = 1, 2, \dots, 5$, $\delta = 0, 1$. Define

$$\beta_{0l\delta} = \int_{\Gamma_{0l\delta}} (V \cdot \nu) ds, \quad (8.3.12)$$

$$(\partial K_{P_0}^*)_- = \left\{ \bigcup_{\beta_{0l\delta} \leq 0} \Gamma_{0l\delta} : 1 \leq l \leq 5, \delta = 0, 1 \right\} \text{ (flow in),}$$

$$(\partial K_{P_0}^*)_+ = \left\{ \bigcup_{\beta_{0l\delta} > 0} \Gamma_{0l\delta} : 1 \leq l \leq 5, \delta = 0, 1 \right\} \text{ (flow out),}$$

$$\beta_{0l\delta}^+ = \max\{\beta_{0l\delta}, 0\}, \quad \beta_{0l\delta}^- = \max\{-\beta_{0l\delta}, 0\}, \quad (8.3.13)$$

and apply the following approximation

$$\int_{\partial K_{P_0}^*} (V \cdot \nu) \Pi_h^* C_h ds \approx \sum_{\substack{1 \leq l \leq 5 \\ \delta = 0, 1}} \{\beta_{0l\delta}^+ C_h(P_0) - \beta_{0l\delta}^- C_h(P_l)\}.$$

Then we have (cf. §7.2)

$$\int_{K_{P_0}^*} \operatorname{div}(V \Pi_h^* C_h) dx dy = \sum_{\substack{1 \leq l \leq 5 \\ \delta = 0, 1}} \{\beta_{0l\delta}^+ C_h(P_0) - \beta_{0l\delta}^- C_h(P_l)\}. \quad (8.3.14)$$

Finally, discretizing the time yields explicit, implicit, or Crank-Nicolson generalized difference schemes. For example, the Crank-Nicolson scheme reads:

$$\begin{aligned}
 & \tau^{-1} \int_{K_{P_0}^*} (C_h^{k+1} - C_h^k) dx dy \\
 = & \frac{1}{2} \int_{K_{P_0}^*} \operatorname{div}(D \operatorname{grad}(C_h^{k+1} + C_h^k)) dx dy \\
 & - \frac{1}{4} \int_{K_{P_0}^*} \operatorname{div}[(V^{k+1} + V^k) \Pi_h^*(C_h^{k+1} + C_h^k)] dx dy \\
 & + \frac{1}{2} \int_{K_{P_0}^*} (I^{k+1} + I^k) dx dy.
 \end{aligned} \tag{8.3.15}$$

The diffusion term on the right-hand side is computed according to (8.3.3) and (8.3.3a-c) with $m = 1$, and the convection term according to (8.3.12) and (8.3.14).

The computation will be simpler if a circumcenter dual grid is adopted. In this case, as in Fig. 8.1.1, we once again integrate the equation (8.3.9) on $K_{P_0}^*$ to obtain an equation similar to (8.3.10). The diffusion term is treated analogously as (8.3.3), but the line integral is computed on the pieces $\overline{Q_i Q_{i+1}}$. In particular when $D = \alpha E$ (a scalar matrix)

$$\begin{aligned}
 & \int_{K_{P_0}^*} \operatorname{div}(\alpha E \operatorname{grad} C_h) dx dy \\
 = & \alpha \int_{\partial K_{P_0}^*} \frac{\partial C_h}{\partial \nu} ds = \alpha \sum_{i=1}^6 \int_{\overline{Q_i Q_{i+1}}} \frac{\partial C_h}{\partial \nu} ds \\
 = & \alpha \sum_{i=1}^6 \frac{C_h(P_{i+1}) - C_h(P_0)}{|\overline{P_0 P_{i+1}}|} \cdot |\overline{Q_i Q_{i+1}}|.
 \end{aligned} \tag{8.3.16}$$

($P_7 = P_1$, $Q_7 = Q_1$.)

The computation of the convection term is similar to (8.3.11). In detail, we have

$$\int_{\partial K_{P_0}^*} (V \cdot \nu) \Pi_h^* C_h ds = \sum_{i=1}^6 \int_{\overline{Q_i Q_{i+1}}} (V \cdot \nu) \Pi_h^* C_h ds. \quad (Q_7 = Q_1.)$$

Set

$$\beta_{i+1} = \int_{Q_i Q_{i+1}} (V \cdot \nu) ds,$$

$$\beta_{i+1}^+ = \max\{\beta_{i+1}, 0\}, \quad \beta_{i+1}^- = \max\{-\beta_{i+1}, 0\}.$$

Then we have

$$\begin{aligned} & \int_{Q_i Q_{i+1}} (V \cdot \nu) \Pi_h^* C_h ds \\ &= \beta_{i+1}^+ C_h(P_0) - \beta_{i+1}^- C_h(P_{i+1}) \\ &= \begin{cases} \beta_{i+1} C_h(P_0), & \text{as } \beta_{i+1} > 0, \quad (\text{flowing out of } K_{P_0}^*), \\ \beta_{i+1} C_h(P_{i+1}), & \text{as } \beta_{i+1} \leq 0, \quad (\text{flowing into } K_{P_0}^*). \end{cases} \end{aligned}$$

This means that on the outer normal direction ν , C_h takes the upwind value. Therefore

$$\int_{K_{P_0}^*} \operatorname{div}(V \Pi_h^* C_h) dx dy = \sum_{i=1}^6 \{\beta_{i+1}^+ C_h(P_0) - \beta_{i+1}^- C_h(P_{i+1})\}, \quad (8.3.17)$$

where $Q_7 = Q_1$ etc.

As pointed out in Chapter 7, in the one-dimensional case, the generalized difference scheme here becomes precisely the usual upwind scheme. If we apply it to (8.3.8), then the oscillation disappears, the density front gets narrower and its position is more accurate.

8.3.3 Upwind weighted multi-element balancing method

Sun [A-40] suggests a kind of weighted upwind difference scheme, referred to as a upwind weighted multi-element balancing method. As before, let $T_h = \{K\}$ be a triangulation, and $T_h^* = \{K_{P_0}^*\}$ a barycenter dual grid. A node P_0 and its adjacent points distribute as in Fig. 8.3.1. Choose any an element $K_{Q_i} = \triangle P_0 P_i P_{i+1}$ with P_0 as a vertex and Q_i as the barycenter. Connect Q_i with the three vertexes to form three smaller triangles: $\triangle P_0 P_i Q_i$, $\triangle P_i P_{i+1} Q_i$, $\triangle P_{i+1} P_0 Q_i$ (cf. Fig. 8.3.3). Denote the value of C at P_l by C_{P_l} ($l = 0, i, i+1$).

Take their weighted average

$$\begin{cases} C_{Q_i} = \omega_0 C_{P_0} + \omega_i C_{P_i} + \omega_{i+1} C_{P_{i+1}}, \\ \omega_0 + \omega_i + \omega_{i+1} = 1, \end{cases} \quad (8.3.18)$$

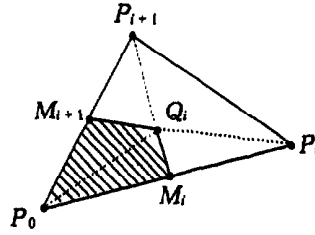


Fig. 8.3.3

where ω_l stands for the upwind weight of P_l and its value is to be determined later on. The restriction on K_{Q_i} of an element $C_h \in U_h$ is a piecewise linear function. More precisely, it is linear respectively on the three sub-triangles, satisfying the interpolation conditions:

$$C_h(P_l) = C_{P_l} \quad (l = 0, i, i+1), \quad C_h(Q_i) = C_{Q_i}.$$

In addition, it satisfies the first boundary condition on the exterior boundary. V_h remains to be the piecewise constant function space, and vanishes at the dual elements where the first boundary condition is given.

Next, let us extend the generalized difference method to the convection-diffusion equation (8.3.9). An integration of this equation on $K_{P_0}^*$ leads to

$$\begin{aligned} \int_{K_{P_0}^*} \frac{\partial C_h}{\partial t} dx dy &= \int_{K_{P_0}^*} \operatorname{div}(D \operatorname{grad} C_h) dx dy \\ &\quad - \int_{K_{P_0}^*} \operatorname{div}(V C_h) dx dy + \int_{K_{P_0}^*} I dx dy. \end{aligned} \quad (8.3.19)$$

For the diffusion term we have ($M_6 = M_1$)

$$\begin{aligned} \int_{K_{P_0}^*} \operatorname{div}(D \operatorname{grad} C_h) dx dy &= \int_{\partial K_{P_0}^*} (D \operatorname{grad} C_h) \cdot \nu ds \\ &= \sum_{i=1}^5 \left\{ \int_{Q_i M_i} (D \operatorname{grad} C_h) \cdot \nu ds + \int_{Q_i M_{i+1}} (D \operatorname{grad} C_h) \cdot \nu ds \right\}. \end{aligned} \quad (8.3.20)$$

For the convection term

$$\begin{aligned} \int_{K_{P_0}^*} \operatorname{div}(V C_h) dx dy &= \int_{\partial K_{P_0}^*} (V \cdot \nu) C_h ds \\ &= \sum_{i=1}^5 \left\{ \int_{Q_i M_i} (V \cdot \nu) C_h ds + \int_{Q_i M_{i+1}} (V \cdot \nu) C_h ds \right\}. \end{aligned} \quad (8.3.21)$$

Expressing C_h on $\triangle P_0 P_i Q_i$ and $\triangle P_0 Q_i P_{i+1}$ in terms of C_{P_0} , C_{P_i} and $C_{P_{i+1}}$, we can obtain an ordinary differential equation of $C_{P_0}(t)$. We may further discretize the time t to obtain explicit or implicit difference schemes. For details, see [A-40].

It is an interesting question how to choose the weight coefficients ω_i . If we take $\omega_i = \frac{1}{3}$, then (8.3.19) is nothing but an ordinary generalized difference scheme without any upwind weighting. Next, we present a method to determine the upwind weighting for each element. Let V be an average velocity vector of the element $\triangle P_0 P_i P_{i+1}$. (cf. Fig. 8.3.3.) For instance, we can set $V = V(Q_i)$. Let $V_{0,i}$, $V_{i,i+1}$ and $V_{i+1,0}$ be projections of V onto $\overline{P_0 P_i}$, $\overline{P_i P_{i+1}}$, $\overline{P_{i+1} P_0}$, respectively. Define the local Peclet number

$$\tau_{0,i} = V_{0,i} |\overline{P_0 P_i}| / (\alpha_i |V|), \quad (8.3.22)$$

where $|V|$ is the length of vector V and α_i is the local diffusion on $\triangle P_0 P_i P_{i+1}$. $\tau_{0,i} > 0$ means that P_0 is on the upwind side of P_i , and $\tau_{0,i} < 0$ the downwind side. Similarly we can define $\tau_{i,i+1}$ and $\tau_{i+1,0}$. Write $\tau_0 = \tau_{0,i} + \tau_{i+1,0}$, indicating, both qualitatively and quantitatively, the upwind or downwind position of P_0 with respect to P_i and P_{i+1} . τ_i and τ_{i+1} can be defined in like manner. Finally we define

$$\omega_0 = \frac{1}{3}(1 + \lambda \tau_0) = \frac{1}{3} + \lambda \frac{V_{0,i} |\overline{P_0 P_i}| - V_{i+1,0} |\overline{P_0 P_{i+1}}|}{\alpha_i |V|}, \quad (8.3.23a)$$

$$\omega_i = \frac{1}{3}(1 + \lambda\tau_i) = \frac{1}{3} + \lambda \frac{V_{i,i+1}|\overline{P_i P_{i+1}}| - V_{0,i}|\overline{P_0 P_i}|}{\alpha_i|V|}, \quad (8.3.23b)$$

$$\omega_{i+1} = \frac{1}{3}(1 + \lambda\tau_{i+1}) = \frac{1}{3} + \lambda \frac{V_{i+1,0}|\overline{P_0 P_{i+1}}| - V_{i,i+1}|\overline{P_i P_{i+1}}|}{\alpha_i|V|}. \quad (8.3.23c)$$

Obviously $\omega_0 + \omega_i + \omega_{i+1} = 1$. Here the coefficient $\lambda > 0$ remains to be chosen. Numerical experiments indicate that λ should be neither too large nor too small. A too large λ will cause extra numerical diffusion, while a too small one will not be good enough to prevent the oscillation of the solution. A proper value of λ might be between 0.004-0.005, as suggested by the numerical experiments.

8.4 Stokes Equation

Consider a Navier-Stokes equation describing an n -dimensional viscous incompressible flow

$$\begin{cases} \frac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla)u + \text{grad} p = f, & (8.4.1a) \\ \text{div} u = 0, & (8.4.1b) \end{cases}$$

where $u = (u_1, \dots, u_n)^T$ is the flow velocity, p the pressure, $\mu > 0$ the viscosity coefficient, and $f = (f_1, \dots, f_n)^T$ the density of the body force. Assume that the velocity u is small enough such that the nonlinear convection term can be ignored, then equation (8.4.1) is reduced to a Stokes equation:

$$\begin{cases} \frac{\partial u}{\partial t} - \mu \Delta u + \text{grad} p = f, \\ \text{div} u = 0. \end{cases}$$

In this section, let $\Omega \subset \mathbb{R}^2$ and restrict our attention to the steady-state case, i.e., $\frac{\partial u}{\partial t} = 0$. So we consider the steady-state Stokes equation:

$$\begin{cases} -\mu \Delta u + \text{grad} p = f, & \text{on } \Omega \subset \mathbb{R}^2, & (8.4.2a) \\ \text{div} u = 0. & & (8.4.2b) \end{cases}$$

Although it is only a linear equation, it has attracted people's attention due to the presence of the incompressible condition. We suppose u satisfies the first boundary condition: $u = 0$ when $x = (x_1, x_2) \in \partial\Omega$. Let $H_0^1(\Omega)$ be a usual Sobolev space, and

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : (q, 1) = \int_{\Omega} q dx = 0 \right\}.$$

Define

$$a(u, v) = \mu \sum_{i,j=1}^2 \left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j} \right),$$

$$b(v, p) = \sum_{i=1}^2 \left(\frac{\partial p}{\partial x_i}, v_i \right), \quad c(v, q) = -(q, \operatorname{div} v).$$

Then, a variational form for (8.4.2) is: Find $(u, p) \in (H_0^1(\Omega))^2 \times L_0^2(\Omega)$ such that

$$\begin{cases} a(u, v) + b(v, p) = (f, v), & \forall v \in (H_0^1(\Omega))^2, \\ c(u, q) = 0, & \forall q \in L_0^2(\Omega). \end{cases} \quad (8.4.3a)$$

Assume that $\Gamma = \partial\Omega$ is Lipschitz continuous, that Ω is a convex domain, and that $f \in (L^2(\Omega))^2$. Then (8.4.3) possesses a unique solution $(u, p) \in (H_0^1(\Omega))^2 \times (H^1(\Omega) \cap L_0^2(\Omega))$, and there is a constant C independent of (u, p) and f such that (see [B-34] for details)

$$\|u\|_{2,\Omega} + \|p\|_{1,\Omega} \leq C \|f\|_{0,\Omega}.$$

8.4.1 Nonconforming generalized difference method

Now let us construct a generalized difference method approximating (8.4.2). Let Ω be a convex polygonal region, and $T_h = \{K_Q\}$ a triangulation of Ω , where K_Q is a triangular element with barycenter Q . Take the midpoints of the sides of the element as nodes. Denote the interior nodes of Ω by $P_1, \dots, P_M$, and the boundary nodes on Γ by $P_{M+1}, \dots, P_N$. The trial function space related to U_h is a piecewise linear function space, with $P_1, \dots, P_N$ as the interpolation nodes and with zero value at boundary nodes. Clearly an element in U_h is not necessarily globally continuous. Corresponding to each interior node

P_i , there is a nodal basis function $\phi_i(x)$ satisfying $\phi_i(P_j) = \delta_{ij}$, $1 \leq i \leq M$, $1 \leq j \leq N$. So every $u_h \in (U_h)^2$ has an expression:

$$u_h(x) = \sum_{i=1}^N u_h(P_i) \phi_i(x), \quad x \in \Omega. \quad (8.4.4)$$

Next we turn to construct the dual grid and the test function space. Let $P_1 \in K_{Q_1} \cap K_{Q_2}$ be an interior node, where $K_{Q_1} = \triangle A_1 A_2 A_3$ and $K_{Q_2} = \triangle A_1 A_2 A_4$. Connect Q_1 and A_1 , Q_1 and A_2 , Q_2 and A_1 , and Q_2 and A_2 respectively to form a tetragon $K_{P_1}^* = \square A_1 Q_2 A_2 Q_1 A_1$ containing P_1 in its interior. This tetragon is called a dual element containing P_1 (cf. Fig. 8.4.1(a)). If P_1 is a boundary node, as shown in Fig. 8.4.1(b), then the dual grid containing P_1 is a triangle $K_{P_1}^* = \triangle A_1 A_2 Q_1$. The entire dual elements constitute a new grid $T_h^* = \{K_{P_i}^*\}$ of Ω , referred to as a dual grid. The test function space related to T_h^* is chosen as the piecewise constant function space on T_h^* , subject to the zero boundary condition. The nodal basis function ψ_j ($1 \leq j \leq M$) is the characteristic function of $K_{P_j}^*$. A function $v_h \in (V_h)^2$ can be expressed as

$$v_h(x) = \sum_{j=1}^M v_h(P_j) \psi_j(x), \quad x \in \Omega. \quad (8.4.5)$$

As before, we use Π_h^* to denote the interpolation projection operator from U_h onto V_h : $\Pi_h^* \phi_i = \psi_i$, $1 \leq i \leq M$.

We also design a subspace W_h for the pressure p_h , which contains all the piecewise constant functions related to T_h , that is, $p_h \in W_h$ equals to a constant $p_h(Q_i)$ on each element K_{Q_i} ($i = 1, 2, \dots, \bar{N}$, $\bar{N}$ being the number of the elements).

Let h_Q and ρ_Q be the maximum length of the sides and the diameter of the inscribed circle of the element K_Q , respectively. We require the grid T_h to be quasi-uniform, i.e., there are constants $\gamma_i > 0$, $i = 0, 1$, such that

$$\gamma_0 h \leq h_Q \leq \gamma_1 \rho_Q, \quad \forall K_Q \in T_h. \quad (8.4.6)$$

The generalized difference scheme given below is nonconforming since $U_h \not\subset H_0^1(\Omega)$. For $u_h \in (U_h)^2$, $v_h \in (V_h)^2$ and $p_h, q_h \in W_h$,

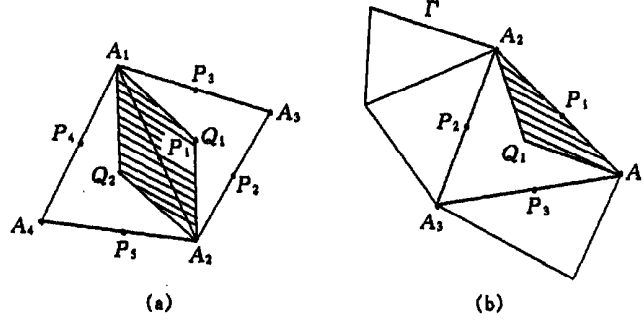


Fig. 8.4.1

set

$$\begin{aligned}
 a(u_h, v_h) &= -\mu \sum_{i=1}^M \int_{K_{P_i}^*} \Delta u_h \cdot v_h dx \\
 &= -\mu \sum_{i=1}^M \int_{\partial K_{P_i}^*} \frac{\partial u_h}{\partial \nu} \cdot v_h ds \\
 &= -\mu \sum_{i=1}^M v_h(P_i) \cdot \int_{\partial K_{P_i}^*} \frac{\partial u_h}{\partial \nu} ds,
 \end{aligned} \tag{8.4.7}$$

$$b(v_h, p_h) = \sum_{i=1}^M v_h(P_i) \cdot \int_{\partial K_{P_i}^*} p_h \nu ds, \tag{8.4.8}$$

$$c(u_h, q_h) = - \sum_{k=1}^N q_h(Q_k) \int_{K_{Q_k}} \operatorname{div} u_h dx, \tag{8.4.9}$$

$$(f, v_h) = \sum_{i=1}^M v_h(P_i) \cdot \int_{K_{P_i}^*} f dx. \tag{8.4.10}$$

Now, we are in a position to introduce a generalized difference method approximating the Stokes equation (8.4.2): Find $(u_h, p_h) \in (U_h)^2 \times W_h$ such that

$$\begin{cases} a(u_h, v_h) + b(v_h, p_h) = (f, v_h), & \forall v_h \in (V_h)^2, \\ c(u_h, q_h) = 0, & \forall q_h \in W_h. \end{cases} \quad \begin{matrix} (8.4.11a) \\ (8.4.11b) \end{matrix}$$

Here the bilinear forms a, b and c are computed according to (8.4.7)-(8.4.9). For instance, we may take $v_h = \begin{pmatrix} \psi_j \\ 0 \end{pmatrix}$ or $v_h = \begin{pmatrix} 0 \\ \psi_j \end{pmatrix}$, and $q_h = \chi_k$ (the characteristic function of K_{Q_k}). (8.4.11) is a linear system of $(2M + \bar{N})$ equations with a (discretized) velocity field at the nodes P_i 's, and a pressure field at the barycenters Q_k 's. So this is a kind of alternative scheme. This system is symmetric since $b(\Pi_h^* u_h, p_h) = c(u_h, p_h)$ as we shall show below.

8.4.2 Convergence and error estimate

For $w_h \in (U_h)^2$, define $D_{x_i h} w_h$ ($i = 1, 2$) as a piecewise function, identical to $\frac{\partial w_h}{\partial x_i}$ in the interior of each element $K_Q \in T_h$. Write

$$(\text{grad}_h u_h, \text{grad}_h w_h) = (D_{x_1 h} u_h, D_{x_1 h} w_h) + (D_{x_2 h} u_h, D_{x_2 h} w_h),$$

$$\|u_h\|_{1h}^2 = (u_h, u_h) + (\text{grad}_h u_h, \text{grad}_h u_h),$$

$$|u_h|_{1,h}^2 = (\text{grad}_h u_h, \text{grad}_h u_h).$$

Lemma 8.4.1

(i) The seminorm $|u_h|_{1h}$ and the norm $\|u_h\|_{1h}$ are equivalent on $(U_h)^2$.

$$\begin{aligned} \text{(ii)} \quad \|u_h\|_0^2 &= \|\Pi_h^* u_h\|_0^2 \\ &= \frac{1}{3} \sum_{K \in T_h} S_K (|u_h(P_1)|^2 + |u_h(P_2)|^2 + |u_h(P_3)|^2), \end{aligned}$$

where S_K is the area of the element K and P_i 's are the midpoints of the three sides of K .

Proof (i) can be proved by the Poincaré inequality (cf. [B-86]). A direct integration of the area coordinate expression of the quadratic polynomial leads to (ii). $\square$

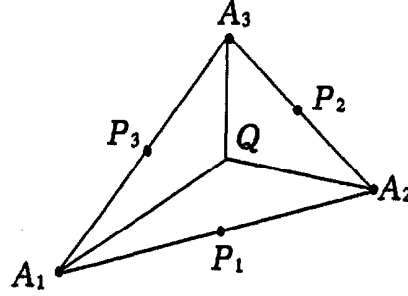


Fig. 8.4.2

Lemma 8.4.2 For $u_h, w_h \in (U_h)^2$ we have constants $C_1, C_2 > 0$ such that

- (i) $a(u_h, \Pi_h^* w_h) = a(w_h, \Pi_h^* u_h)$;
- (ii) $|a(u_h, \Pi_h^* w_h)| \leq C_1 \|u_h\|_{1h} |w_h|_{1h}$;
- (iii) $a(u_h, \Pi_h^* u_h) \geq C_2 |u_h|_{1h}^2$.

Proof Consider an element $K = K_Q$. As in Fig. 8.4.2 we have

$$a(u_h, \Pi_h^* w_h) = \sum_{K \in T_h} I_K,$$

$$\begin{aligned}
 I_K &= -w_h(P_1) \left(\int_{A_2 Q A_1} \frac{\partial u_h}{\partial x_1} dx_2 - \int_{A_2 Q A_1} \frac{\partial u_h}{\partial x_2} dx_1 \right) \\
 &\quad - w_h(P_2) \left(\int_{A_3 Q A_2} \frac{\partial u_h}{\partial x_1} dx_2 - \int_{A_3 Q A_2} \frac{\partial u_h}{\partial x_2} dx_1 \right) \\
 &\quad - w_h(P_3) \left(\int_{A_1 Q A_3} \frac{\partial u_h}{\partial x_1} dx_2 - \int_{A_1 Q A_3} \frac{\partial u_h}{\partial x_2} dx_1 \right) \\
 &= -\frac{\partial u_h}{\partial x_1} [w_h(P_1)(x_2(A_1) - x_2(A_2)) \\
 &\quad + w_h(P_2)(x_2(A_2) - x_2(A_3)) + w_h(P_3)(x_2(A_3) - x_2(A_1))]
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\partial u_h}{\partial x_2} [w_h(P_1)(x_1(A_1) - x_1(A_2)) \\
& + w_h(P_2)(x_1(A_2) - x_1(A_3)) + w_h(P_3)(x_1(A_3) - x_1(A_1))].
\end{aligned} \tag{8.4.12}$$

Let $(\lambda_1, \lambda_2, \lambda_3)$ be the coordinates of the barycenter of K , and $\mu_i = \lambda_i + \lambda_{i+1} - \lambda_{i+2}$ for $i = 1, 2, 3$ ($\lambda_4 = \lambda_1, \lambda_5 = \lambda_2$). Then we have $\mu_i(P_j) = \delta_{ij}$ and

$$w_h|_K = w_h(P_1)\mu_1 + w_h(P_2)\mu_2 + w_h(P_3)\mu_3, \tag{8.4.13}$$

$$\begin{cases} \frac{\partial \lambda_i}{\partial x_1} = \frac{x_2(A_{i+1}) - x_2(A_{i+2})}{2S_K}, \\ \frac{\partial \lambda_i}{\partial x_2} = \frac{x_1(A_{i+1}) - x_1(A_{i+2})}{2S_K}, \end{cases} \tag{8.4.14}$$

where $i = 1, 2, 3$ and $A_4 = A_1, A_5 = A_2$. Substituting (8.4.13) and (8.4.14) into (8.4.12) yields

$$I_K = \int_K \left(\frac{\partial u_h}{\partial x_1} \frac{\partial w_h}{\partial x_1} + \frac{\partial u_h}{\partial x_2} \frac{\partial w_h}{\partial x_2} \right) dx.$$

Thus

$$a(u_h, \Pi_h^* w_h) = \sum_{K \in T_h} \int_K \text{grad} u_h \cdot \text{grad} w_h dx = (\text{grad}_h u_h, \text{grad}_h w_h).$$

This implies (i), (ii), (iii) and completes the proof. $\square$

The following Lemma can be proved in like manner.

Lemma 8.4.3 For $u_h \in (U_h)^2$ and $p_h \in W_h$ we have ($C_3 > 0$ is a constant)

$$(i) \quad b(\Pi_h^* u_h, p_h) = c(u_h, p_h), \tag{8.4.15}$$

$$(ii) \quad |b(\Pi_h^* u_h, p_h)| \leq C_3 |u_h|_{1h} |p_h|_0. \tag{8.4.16}$$

For $u \in (H_0^1(\Omega))^2$, define its projection onto $(U_h)^2$ (cf. Fig. 8.4.2) by

$$\hat{\Pi}_h u = \hat{u}(P_1)\mu_1 + \hat{u}(P_2)\mu_2 + \hat{u}(P_3)\mu_3, \text{ in each } K \in T_h,$$

$$\hat{u}(P_i) = \frac{1}{|\overline{A_i A_{i+1}}|} \int_{\overline{A_i A_{i+1}}} u(x) ds, \quad i = 1, 2, 3. \quad (A_4 = A_1.)$$

Obviously $\hat{\Pi}_h u|_K = u$, $\forall K \in T_h$ when $u \in (U_h)^2$. So it follows from the interpolation approximation property that

$$|\hat{\Pi}_h u|_{1h} \leq C_4 |u|_1.$$

Lemma 8.4.4

(i) It holds for $u \in (H_0^1(\Omega) \cap C^2(\overline{\Omega}))^2$, and $u_h \in (U_h)^2$ that

$$|a(u - \hat{\Pi}_h u, \Pi_h^* u_h)| \leq C_5 h |u_h|_{1h} |D^2 u|_{\max}. \quad (8.4.17)$$

(ii) For $p \in C^1(\overline{\Omega})$, $u_h \in (U_h)^2$ and $p_h \in W_h$ we have

$$|b(\Pi_h^* u_h, p - p_h)| \leq C_6 \max_{\Omega} |p - p_h| |u_h|_{1h}. \quad (8.4.18)$$

(iii) If $u \in (H_0^1(\Omega))^2$, then

$$c(u - \hat{\Pi}_h u, q_h) = 0, \quad \forall q_h \in W_h. \quad (8.4.19)$$

Here $|D^2 u|_{\max}$ stands for the maximum norm of the second partial derivatives of u . Moreover, the equality (8.4.19) is equivalent to

$$\sum_{k=1}^{\tilde{N}} \int_{K_{Q_k}} q_h \operatorname{div}(u - \hat{\Pi}_h u) dx = 0. \quad (8.4.20)$$

Proof (8.4.17) and (8.4.18) are direct consequences of (8.4.7), (8.4.8), and the expressions of $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$. To show (8.4.19), we denote by ν_1, ν_2 and ν_3 the unit outer normal directions of $\overline{A_1 A_2}, \overline{A_2 A_3}$ and $\overline{A_3 A_1}$ respectively, then

$$\begin{aligned} c(u - \hat{\Pi}_h u, q_h) &= - \sum_{k=1}^{\tilde{N}} q_h(Q_k) \sum_{i=1}^3 \nu_i \int_{\overline{A_i A_{i+1}}} (u - \hat{\Pi}_h u) ds \\ &= \sum_{k=1}^{\tilde{N}} q_h(Q_k) \sum_{i=1}^3 \nu_i \left[\int_{\overline{A_i A_{i+1}}} u ds - |\overline{A_i A_{i+1}}| (\hat{\Pi}_h u)(P_i) \right] = 0. \end{aligned}$$

Finally, the equivalence of (8.4.20) and (8.4.19) follows from (8.4.9) and the fact $\operatorname{div} \hat{\Pi}_h u \in W_h$. This completes the proof. $\square$

It results from (8.4.15) and (8.4.20) that there is a constant $\gamma > 0$ satisfying

$$\sup \frac{b(\Pi_h^* u_h, q_h)}{|u_h|_{1h}} \geq \gamma \|q_h\|_0, \quad \forall q_h \in W_h. \quad (8.4.21)$$

Making use of Lemmas 8.4.1-4, one may easily prove (cf. [B-34,86]) the following: If $(u, p) \in (C^2(\bar{\Omega}))^2 \cap (H_0^1(\Omega))^2 \times C^1(\bar{\Omega}) \cap L_0^2(\Omega)$ is the solution to (8.4.3), then, for sufficiently small $h > 0$, the nonconforming generalized difference equation (8.4.11) has a unique solution $(u_h, p_h) \in (U_h)^2 \times W_h$, and

$$|u - u_h|_{1h} + \|p - p_h\|_0 \leq Ch(|D^2 u|_{\max} + |Dp|_{\max}),$$

where $|Dp|_{\max}$ denotes the maximum norm of the first derivative of p .

8.4.3 A numerical example

In the Stokes problem (8.4.2), take $\Omega = [0, 1] \times [0, 1]$, $\mu = 1$, $f = (f_1, f_2)$:

$$\begin{aligned} f_1(x, y) &= -6x^2(x-1)^2(2y-1), \\ f_2(x, y) &= 4x(x-1)(2x-1)[(2x-1)^2 + 2y(y-1)] - f_1(y, x). \end{aligned}$$

Its true solution is

$$\begin{aligned} u_1(x, y) &= x^2(x-1)^2y(y-1)(2y-1), \\ u_2(x, y) &= -u_1(y, x), \\ p(x, y) &= 2x(x-1)(2x-1)y(y-1)(2y-1). \end{aligned}$$

Divide Ω into thirty-six small squares with side size $\frac{1}{6}$. Then we use the diagonal lines at an angle of $\pi/4$ to the x -axis to further divide each small square into two triangles, ending up with a triangulation T_h . The errors of the approximation is given below:

$$\begin{aligned} \max_j |u_h(P_j) - u(P_j)| &= 0.31616975 \times 10^{-2}, \\ \min_j |u_h(P_j) - u(P_j)| &= 0.80926718 \times 10^{-5}, \end{aligned}$$

$$\begin{aligned}\max_Q |p_h(Q) - p(Q)| &= 0.14746220 \times 10^{-1}, \\ \min_Q |p_h(Q) - p(Q)| &= 0.56510426 \times 10^{-3}.\end{aligned}$$

8.5 Coupled Sound-Heat Problems

The following hyperbolic-parabolic system describes the flow of compressible fluid with heat transfer:

$$\begin{cases} \frac{\partial u}{\partial t} = c \frac{\partial}{\partial x} (w - (\gamma - 1)e), \\ \frac{\partial w}{\partial t} = c \frac{\partial u}{\partial x}, \\ \frac{\partial e}{\partial t} = \sigma \frac{\partial^2 e}{\partial x^2} - c \frac{\partial u}{\partial x}, \end{cases} \quad 0 \leq x \leq 1, 0 < t \leq T, \quad (8.5.1)$$

where c and σ are positive constants, and $\gamma > 1$. The initial values

$$u(x, 0) = f(x), \quad w(x, 0) = g(x), \quad e(x, 0) = h(x)$$

are 1-periodic functions, and the boundary condition is a 1-periodic boundary condition. Write $I = [0, 1]$ and $V = \{v \in H^1(I) : v(0) = v(1)\}$. Then, a weak form of (8.5.1) is: Find $u, w, e \in V$ such that for all $v \in V$

$$\begin{cases} (u_t, v) - c(w_x, v) + c(\gamma - 1)(e_x, v) = 0, \\ (w_t, v) - c(u_x, v) = 0, \\ (e_t, v) + c(u_x, v) + \sigma(e_x, v_x) = 0. \end{cases} \quad (8.5.2)$$

Decompose $I = [0, 1]$ by $0 = x_0 < x_1 < \dots < x_N = 1$, and set $I_i = [x_{i-1}, x_i]$, $h_i = x_i - x_{i-1}$, then we have a grid $T_h = \{I_i\}$. The dual grid $T_h^* = \{I_i^*\}$, where $I_i^* = [x_{i-1/2}, x_{i+1/2}]$, $x_{i-1/2} = \frac{1}{2}(x_{i-1} + x_i)$, for $i = 1, 2, \dots, N-1$, and $I_0^* = [0, x_{1/2}]$, $I_N^* = [x_{N-1/2}, x_N]$. The trial function space U_h related to T_h consists of the piecewise linear continuous functions with period 1. For $i = 1, 2, \dots, N-1$, the nodal basis functions $\phi_i(x)$'s are defined by

$$\phi_i(x_j) = \delta_{ij}, \quad 0 \leq j \leq N,$$

and $\phi_N(x)$ is defined by

$$\phi_N(x_0) = 1, \phi_N(x_j) = \delta_{Nj}, 1 \leq j \leq N.$$

The test function space V_h is composed of the piecewise constant periodic function on T_h^* with the nodal basis function $\psi_j(x)$ ($1 \leq j \leq N$) being the characteristic function on I_j^* . Any $u_h, w_h, e_h \in U_h$ can be expressed as

$$u_h(x, t) = \sum_{i=1}^N u_i(t) \phi_i(x),$$

$$w_h(x, t) = \sum_{i=1}^N w_i(t) \phi_i(x),$$

$$e_h(x, t) = \sum_{i=1}^N e_i(t) \phi_i(x).$$

Now, a generalized difference scheme approximating (8.5.2) is: Find $u_h, w_h, e_h \in U_h$ such that for $1 \leq j \leq N$

$$\begin{cases} (u_{ht}, \psi_j) - c(w_{hx}, \psi_j) + c(\gamma - 1)(e_{hx}, \psi_j) = 0, \\ (w_{ht}, \psi_j) - c(u_{hx}, \psi_j) = 0, \\ (e_{ht}, \psi_j) + c(u_{hx}, \psi_j) + \sigma(e_{hx}, \psi_j) = 0. \end{cases} \quad (8.5.3)$$

Write $v_h(t) = (u_1(t), \dots, u_N(t), w_1(t), \dots, w_N(t), e_1(t), \dots, e_N(t))^T$, then one can rewrite (8.5.3) as

$$M \frac{dv_h(t)}{dt} - K v_h(t) = 0, \quad (8.5.4)$$

where

$$M = \begin{bmatrix} M_1 & & \\ & M_1 & \\ & & M_1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & cK_1 & -c(\gamma - 1)K_1 \\ cK_1 & 0 & 0 \\ -cK_1 & 0 & \sigma K_2 \end{bmatrix}$$

are $(3N \times 3N)$ matrices, and M_1, K_1, K_2 are $(N \times N)$ matrices: The entry at the j -th row and i -th column of M_1 is (ϕ_i, ϕ_j) , the one of K_1

is (ϕ_{ix}, ψ_j) , and the one of K_2 is (ϕ_{ix}, ψ_{jx}) (in the sense of generalized functions). The initial value is

$$v_h(0) = (f(x_1), \dots, f(x_N), g(x_1), \dots, g(x_N), h(x_1), \dots, h(x_N))^T.$$

(8.5.4) is a system of ordinary differential equations. Various kinds of numerical methods for ordinary differential equations can be used here for a further discretization. For example, the modified Euler's method gives a fully-discretized generalized difference scheme:

$$M \frac{v_h^{n+1} - v_h^n}{\tau} = K \frac{v_h^{n+1} + v_h^n}{2}. \quad (8.5.5)$$

This is a six-point symmetric scheme (Crank-Nicolson scheme). If the solution to (8.1.1) is smooth, and the step size $h_i = h$, then the truncation error of (8.5.5) is $O(\tau^2 + h^2)$. In this case (8.5.5) can be written as

$$\begin{aligned} & u_{j-1}^{n+1} + 6u_j^{n+1} + u_{j+1}^{n+1} + \frac{2c\tau}{h}(w_{j-1}^{n+1} - w_{j+1}^{n+1}) \\ & - \frac{2c(\gamma-1)\tau}{h}(e_{j-1}^{n+1} - e_{j+1}^{n+1}) \\ = & u_{j-1}^n + 6u_j^n + u_{j+1}^n + \frac{2c\tau}{h}(w_{j+1}^n - w_{j-1}^n) \\ & - \frac{2c(\gamma-1)\tau}{h}(e_{j+1}^n - e_{j-1}^n), \end{aligned} \quad (8.5.6a)$$

$$\begin{aligned} & \frac{2c\tau}{h}(u_{j-1}^{n+1} - u_{j+1}^{n+1}) + w_{j-1}^{n+1} + 6w_j^{n+1} + w_{j+1}^{n+1} \\ = & \frac{2c\tau}{h}(u_{j+1}^n - u_{j-1}^n) + w_{j-1}^n + 6w_j^n + w_{j+1}^n, \end{aligned} \quad (8.5.6b)$$

$$\begin{aligned} & \frac{2c\tau}{h}(u_{j+1}^{n+1} - u_{j-1}^{n+1}) + \left(1 - \frac{4\sigma\tau}{h^2}\right)e_{j-1}^{n+1} \\ & + \left(6 + \frac{8\sigma\tau}{h^2}\right)e_j^{n+1} + \left(1 - \frac{4\sigma\tau}{h^2}\right)e_{j+1}^{n+1} \\ = & \frac{2c\tau}{h}(u_{j-1}^n - u_{j+1}^n) + \left(1 + \frac{4\sigma\tau}{h^2}\right)e_{j-1}^n \\ & + \left(6 - \frac{8\sigma\tau}{h^2}\right)e_j^n + \left(1 + \frac{4\sigma\tau}{h^2}\right)e_{j+1}^n, \end{aligned} \quad (8.5.6c)$$

where

$$u_k = u_{N+k}, w_k = w_{N+k}, e_k = e_{N+k}, k = 0, 1, \dots$$

If we use Euler's method in place of (8.5.6a,b), while keeping (8.5.6c) as it is, then (8.5.6a,b) are replaced by

$$\begin{aligned} & u_{j-1}^{n+1} + 6u_j^{n+1} + u_{j+1}^{n+1} \\ = & u_{j-1}^n + 6u_j^n + u_{j+1}^n - \frac{4c\tau}{h}(w_{j-1}^n - w_{j+1}^n) \\ & - \frac{4c\tau(\gamma-1)}{h}(e_{j+1}^n - e_{j-1}^n), \end{aligned} \quad (8.5.6a)'$$

$$\begin{aligned} & w_{j-1}^{n+1} + 6w_j^{n+1} + w_{j+1}^{n+1} \\ = & -\frac{4c\tau}{h}(u_{j+1}^n - u_{j-1}^n) + w_{j-1}^n + 6w_j^n + w_{j+1}^n. \end{aligned} \quad (8.5.6b)'$$

Its truncation error is $O(\tau+h^2)$. This scheme is implicit, but (8.5.6a, b) are both triple diagonal matrices for the unknowns $\{u_j^{n+1}\}$ and $\{w_j^{n+1}\}$ respectively. If we work out $\{u_j^{n+1}\}$ and insert it into (8.5.6c), then (8.5.6c) also becomes a triple diagonal matrix for $\{e_j^{n+1}\}$, which is easy to solve numerically.

Next we try to investigate the stability of (8.5.6) as with the usual difference methods. By the separation of variables it is easy to deduce the amplification matrix of (8.5.6):

$$G(\xi, h) = (a^2d - b^2d - (\gamma-1)ab^2)^{-1}.$$

$$\begin{bmatrix} a^2d + b^2d + (\gamma-1)ab^2 & 2abcd & -2(\gamma-1)a^2b \\ 2abd & a^2d + b^2d - (\gamma-1)ab^2 & -2(\gamma-1)ab^2 \\ -2a^2b & -2ab^2 & (\gamma-1)ab^2 + a^2l - b^2l \end{bmatrix}$$

where

$$a = 2(3 + \cos \xi), \quad b = i \frac{4rhc}{\sigma} \sin \xi, \quad (i = \sqrt{-1})$$

$$d = 2(3 + 4r) + 2(1 - 4r) \cos \xi = 4 \left(2 - (1 - 4r) \sin^2 \frac{\xi}{2} \right),$$

$$l = 2(3 - 4r) + 2(1 + 4r) \cos \xi = 4 \left(2 - (1 + 4r) \sin^2 \frac{\xi}{2} \right),$$

$$r = \frac{\sigma\tau}{h^2} = \text{constant}.$$

Since $a \geq 4$, $d \geq 4$ and $b^2 = O(h^2) = O(\tau)$, we have

$$\begin{aligned} G(\xi, h) &= \begin{bmatrix} 1 & \frac{2b}{a} & \frac{-2(\gamma-1)b}{d} \\ \frac{2b}{a} & 1 & 0 \\ -\frac{2b}{d} & 0 & \frac{l}{d} \end{bmatrix} + O(\tau) \\ &= G_0(\xi, h) + O(\tau). \end{aligned}$$

By [A-27] and [B-74], the matrix family $\{G^n(\xi, h)\}$ ($0 < \tau \leq \tau_0$, $|\xi| \leq \pi$, $0 < n\tau \leq T$) is uniformly bounded, if and only if $\{G_0^n(\xi, h)\}$ ($0 < \tau \leq \tau_0$, $|\xi| \leq \pi$, $0 < n\tau \leq T$) is uniformly bounded. Similarity transformations are employed in [A-56] to show the uniform boundedness of $\{G_0^n(\xi, h)\}$, and hence scheme (8.5.6) is stable for any grid ratio $r > 0$. This together with the consistency of the scheme (truncation error $O(\tau^2 + h^2)$) guarantees the convergence of (u_h, w_h, e_h) to (u, w, e) with an error estimate:

$$\begin{aligned} &\max_{0 \leq t \leq T} \|u(\cdot, t) - u_h(\cdot, t)\|_0 + \max_{0 \leq t \leq T} \|w(\cdot, t) - w_h(\cdot, t)\|_0 \\ &\quad + \max_{0 \leq t \leq T} \|e(\cdot, t) - e_h(\cdot, t)\|_0 \leq C(\tau^2 + h^2). \end{aligned}$$

(cf. [A-56] and [B-74] for details.)

Similarly one can show that the scheme (8.5.6a)', (8.5.6b)', (8.5.6c) is absolutely stable, and its convergence order is $O(\tau + h^2)$.

Remark This section considers only a linear element generalized difference scheme for one-dimensional problems. In principle, high order element schemes can be constructed for triangulations in higher dimensions. But we point out that the discussions of this section on convergence and stability are analogous to that of standard finite difference methods, which can not be extended directly to arbitrary triangulations and high order elements. The equation (8.5.1)

is a coupled hyperbolic-parabolic system. It remains to be an open question how to extend the methods in Chapters 5 and 6 to study the convergence and the error estimate of the generalized difference scheme approximating (8.5.1).

8.6 Regularized Long Wave Equations

This section is devoted to the generalized difference solutions of the following initial and boundary values problem of a regularized long wave equation:

$$\begin{cases} \frac{\partial u}{\partial t} - \gamma \frac{\partial^2}{\partial x^2} \frac{\partial u}{\partial t} = \frac{\partial f(u)}{\partial x}, & a < x < b, 0 < t \leq T, & (8.6.1a) \\ u(a, t) = u(b, t) = 0, & 0 < t \leq T, & (8.6.1b) \\ u(x, 0) = u_0(x), & a < x < b, & (8.6.1c) \end{cases}$$

where $f(u) = \alpha u + \frac{1}{2}\beta u^2$; $\gamma > 0$ and α, β are given constants; $u_0(x) \in C^2(I)$, $I = [a, b]$, $u_0(a) = u_0(b) = 0$. It is confirmed in [B-7] that problem (8.6.1) has a unique solution $u(x, t) \in C^1([0, T]; C^2(I))$. Define

$$a(u, v) = \left(\gamma \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \right) + (u, v) = \gamma \int_a^b u_x v_x dx + \int_a^b u v dx. \quad (8.6.2)$$

Then a weak form of (8.6.1) reads: Find $u(x, t) \in C^1([0, T]; H_0^1(I))$ such that

$$\begin{cases} a\left(\frac{\partial u}{\partial t}, v\right) = \left(\frac{\partial f(u)}{\partial x}, v\right), & \forall v \in H_0^1(I), & (8.6.3a) \\ (u(x, 0), v) = (u_0(x), v), & \forall v \in H_0^1(I). & (8.6.3b) \end{cases}$$

Next we shall present a Lagrange quadratic generalized difference scheme for (8.6.1) (or (8.6.3)).

8.6.1 Semi-discrete generalized difference schemes

Place a grid $T_h = \{I_i\}_1^N$ on $I = [a, b]$, where $I_i = [x_{i-1}, x_i]$, $a = x_0 < x_1 < \dots < x_N = b$. Set $h_i = x_i - x_{i-1}$, $h = \max h_i$. Piecewise quadratic polynomials vanishing on the boundary $x = a$ and b are

chosen as the trial function space $U_h \subset H_0^1(I)$. On each I_i , a function $p \in U_h$ is determined uniquely by its values at three interpolation nodes x_{j-1} , $x_{j-1/2}$ and x_j . The nodal basis functions are (cf. §2.3)

$$\phi_j(x) = \begin{cases} 2h_j^{-2}(x - x_{j-1})(x - x_{j-1/2}), & x_{j-1} \leq x \leq x_j, \\ 2h_{j+1}^{-2}(x - x_{j+1})(x - x_{j+1/2}), & x_j \leq x \leq x_{j+1}, \\ 0, & \text{elsewhere.} \end{cases}$$

$$\phi_{j-1/2}(x) = \begin{cases} 4h_j^{-2}(x - x_{j-1})(x_j - x), & x_{j-1} \leq x \leq x_j, \\ 0, & \text{elsewhere.} \end{cases}$$

The dual grid $T_h^* = \{I_j^*, I_{j-1/2}^*\}$, where $I_j^* = [x_{j-1/4}, x_{j+1/4}]$, $I_{j-1/2}^* = [x_{j-3/4}, x_{j-1/4}]$. The corresponding test function space V_h contains piecewise constant functions, which vanish on $I_0^* = [a, x_{1/4}]$ and $I_N^* = [x_{N-1/4}, b]$. The nodal basis functions ψ_j and $\psi_{j-1/2}$ are the characteristic functions of I_j^* and $I_{j-1/2}^*$ respectively. The semi-discrete generalized difference scheme approximating (8.6.1) is: Find

$$u_h(x, t) = \sum_{j=1}^{N-1} u_j(t) \phi_j(x) + \sum_{j=1}^N u_{j-1/2}(t) \phi_{j-1/2}(x) \in U_h$$

such that

$$\begin{cases} a\left(\frac{\partial u_h}{\partial t}, v_h\right) = \left(\frac{\partial f(u_h)}{\partial x}, v_h\right), & \forall v_h \in V_h, 0 < t \leq T, \\ u_h(x, 0) = u_{0h}(x), & a \leq x \leq b. \end{cases} \quad (8.6.4a)$$

Here u_{0h} is a certain approximation of $u_0(x)$, generally taken as the interpolation projection of $u_0(x)$ onto U_h . Now the above bilinear form $a(\frac{\partial u_h}{\partial t}, v_h)$ should be understood in the sense of generalized functions. (8.6.4) is an initial value problem of a system of ordinary differential equations.

We observe that $u_j(t) = u_h(x_j, t)$ and $u_{j-1/2}(t) = u_h(x_{j-1/2}, t)$. Denote by Π_h^* the interpolation projection from U_h onto V_h . Then, for any $u_h \in U_h$

$$\Pi_h^* u_h = \sum_{j=1}^{N-1} u_j \psi_j + \sum_{j=1}^N u_{j-1/2} \psi_{j-1/2}.$$

Let us introduce the following discrete norms:

$$|u_h|_{0h}^2 = |\Pi_h^* u_h|_0^2 = \sum_{j=1}^N \frac{h_j}{4} (u_{j-1}^2 + 2u_{j-1/2}^2 + u_j^2),$$

$$|u_h|_{1h}^2 = \sum_{j=1}^N \frac{h_j}{2} \left[\left(\frac{u_{j-1/2} - u_{j-1}}{h_j/2} \right)^2 + \left(\frac{u_j - u_{j-1/2}}{h_j/2} \right)^2 \right].$$

As in §2.3, it is easy to show that

(i) On U_h , $|\cdot|_{0h}$ and $|\cdot|_{1h}$ are equivalent to $|\cdot|_0$ and $|\cdot|_1$ respectively;

(ii) $|a(u_h, \Pi_h^* w_h)| \leq M \|u_h\|_1 \|w_h\|_1, \quad \forall w_h, u_h \in U_h;$

(iii) $a(u_h, \Pi_h^* u_h) > \alpha \|u_h\|_1^2, \quad \forall u_h \in U_h,$

where $M > 0$ and $\alpha > 0$ are constants. By virtue of these observations, the following statement holds:

(iv) If $u \in H_0^1(I) \cap H^3(I)$ is the solution to

$$a(u, v) = (g, v), \quad \forall v \in H_0^1(I),$$

and $u_h \in U_h$ is the solution to

$$a(u_h, v_h) = (g, v_h), \quad \forall v_h \in V_h,$$

then we have the following estimate

$$\|u - u_h\|_1 \leq C_1 h^2 |u|_3. \quad (8.6.5)$$

Here and below, C_j 's ($1 \leq j \leq 6$) denote constants independent of h . Now let us consider the unique-solvability and the convergence of the semi-discrete scheme (8.6.4). Define

$$\underline{m} = \inf_{\substack{a \leq x \leq b \\ 0 \leq t \leq T}} u(x, t), \quad \overline{m} = \sup_{\substack{a \leq x \leq b \\ 0 \leq t \leq T}} u(x, t).$$

Take a bounded truncation function $\hat{f}(v) \in C^2(\mathbb{R})$ of $f(v)$ satisfying ($\epsilon > 0$ is a constant)

$$\hat{f}(v) = f(v), \quad \forall v \in [\underline{m} - \epsilon, \overline{m} + \epsilon],$$

$$\sup_{v \in \mathbb{R}} \{|\hat{f}(v)|, |\hat{f}'(v)|, |\hat{f}''(v)|\} = d_\epsilon < \infty.$$

Replacing f by $\hat{f}$ yields a continuous problem related to (8.6.1)

$$\begin{cases} \frac{\partial \hat{u}}{\partial t} - \gamma \frac{\partial^2}{\partial x^2} \frac{\partial \hat{u}}{\partial t} = \frac{\partial \hat{f}(\hat{u})}{\partial x}, & a < x < b, 0 < t \leq T, \end{cases} \quad (8.6.6a)$$

$$\begin{cases} \hat{u}(a, t) = \hat{u}(b, t) = 0, & 0 < t \leq T, \end{cases} \quad (8.6.6b)$$

$$\begin{cases} \hat{u}(x, 0) = u_0(x), & a < x < b, \end{cases} \quad (8.6.6c)$$

and a semi-discrete problem

$$\begin{cases} a \left(\frac{\partial \hat{u}_h}{\partial t}, v_h \right) = \left(\frac{\partial \hat{f}(\hat{u}_h)}{\partial x}, v_h \right), & \forall v_h \in V_h, 0 < t \leq T, \end{cases} \quad (8.6.7a)$$

$$\begin{cases} \hat{u}_h(x, 0) = u_{0h}(x), & a \leq x \leq b. \end{cases} \quad (8.6.7b)$$

Lemma 8.6.1 *The unique solution $u(x, t)$ of problem (8.6.1) is identical to the unique solution of problem (8.6.6).*

Proof The solution $u(x, t)$ of (8.6.1) obviously solves (8.6.6). To show the uniqueness, let $\hat{u}(x, t)$ be any solution to (8.6.6). Then

$$\frac{\partial(u - \hat{u})}{\partial t} - \gamma \frac{\partial^2}{\partial x^2} \frac{\partial(u - \hat{u})}{\partial t} = \frac{\partial \hat{f}(u)}{\partial x} - \frac{\partial \hat{f}(\hat{u})}{\partial x}.$$

Multiply it by $(u - \hat{u})$ and then integrate it for x to obtain

$$\frac{1}{2} \frac{d}{dt} (|u - \hat{u}|_0^2 + \gamma |u - \hat{u}|_1^2) \leq d_\epsilon |u - \hat{u}|_0 |u - \hat{u}|_1,$$

and hence

$$\frac{d}{dt} (|u - \hat{u}|_0^2 + \gamma |u - \hat{u}|_1^2) \leq d'_\epsilon (|u - \hat{u}|_0^2 + \gamma |u - \hat{u}|_1^2),$$

where $d'_\epsilon = d_\epsilon \max\{1, \frac{1}{\gamma}\}$. Solving the last inequality gives

$$\begin{aligned} & |u(\cdot, t) - \hat{u}(\cdot, t)|_0^2 + \gamma |u(\cdot, t) - \hat{u}(\cdot, t)|_1^2 \\ & \leq \exp(d'_\epsilon T) (|u(\cdot, 0) - \hat{u}(\cdot, 0)|_0^2 + \gamma |u(\cdot, 0) - \hat{u}(\cdot, 0)|_1^2) = 0, \end{aligned}$$

which implies $u \equiv \hat{u}$. This completes the proof. $\square$

Lemma 8.6.2 *Problem (8.6.7) has a unique solution $\hat{u}_h(\cdot, t) \in U_h$ defined on $[0, T]$.*

Proof By the existence and uniqueness theorem of the solution to ordinary differential equations, equation (8.6.7) has a unique solution $\hat{u}_h(\cdot, t)$ at least on a right-hand neighborhood of $t = 0$. According to the continuation theorem, to extend the solution $\hat{u}_h(\cdot, t)$ to the entire $[0, T]$ we only have to show the boundedness of $\|\hat{u}_h(\cdot, t)\|_1$ for all such $t \in [0, T]$ where $\hat{u}_h(\cdot, t)$ is well-defined. So we integrate (8.6.7a) with respect to t and make use of (8.6.7b) to get

$$a(\hat{u}_h, v_h) = a(u_{0h}, v_h) + \int_0^t \left(\frac{\partial \hat{f}(\hat{u}_h(\cdot, s))}{\partial x}, v_h(\cdot) \right) ds.$$

Choose $v_h = \Pi_h^* \hat{u}_h$, then it follows from (i), (ii) and (iii) that

$$\|\hat{u}_h(\cdot, t)\|_1 \leq \frac{M}{\alpha} \|u_{0h}\|_1 + \frac{d_\epsilon}{\alpha C_2} \int_0^t \|\hat{u}_h(\cdot, s)\|_1 ds.$$

By virtue of Gronwall's inequality

$$\|\hat{u}_h(\cdot, t)\|_1 \leq \frac{M}{\alpha} \exp(d_\epsilon'' T) \|u_{0h}\|_1.$$

Here $d_\epsilon'' = d_\epsilon/(\alpha C_2)$.

After these preparations, we are able to use a method similar to that in §5.1 to obtain

$$\begin{aligned} & \|u(\cdot, t) - \hat{u}_h(\cdot, t)\|_1 \\ & \leq C_3 \left[\|u_0 - u_{0h}\|_1 + h^2(|u_0|_3 + \int_0^t \left| \frac{\partial u(\cdot, s)}{\partial s} \right|_3 ds) \right]. \end{aligned}$$

Finally, it should be pointed out that, for sufficiently small h , the above $\hat{u}_h(x, t)$ is precisely the solution $u_h(x, t)$ of the difference equation (8.6.4). In fact, by the imbedding theorem and the above estimate we have

$$\begin{aligned} & \sup_{\substack{0 \leq x \leq b \\ 0 \leq t \leq T}} |u(x, t) - \hat{u}_h(x, t)| \\ & \leq C_4 \sup_{0 \leq t \leq T} \|u(\cdot, t) - \hat{u}_h(\cdot, t)\|_1 \rightarrow 0, \text{ as } h \rightarrow 0. \end{aligned}$$

Thus for $h > 0$ sufficiently small

$$\underline{m} - \epsilon \leq \hat{u}_h(x, t) \leq \overline{m} + \epsilon, \quad a \leq x \leq b, \quad 0 \leq t \leq T.$$

Consequently

$$\hat{f}(\hat{u}_h(x, t)) = f(u_h(x, t)).$$

This implies $\hat{u}_h(x, t) = u_h(x, t)$.

To sum up, we have

Theorem 8.6.1 *If the solution $u(x, t)$ of the problem (8.6.1) satisfies $u(x, t) \in C^1([0, T]; H^3(I))$, then for sufficiently small $h > 0$, the semi-discrete scheme (8.6.4) possesses a unique solution $u_h(\cdot, t) \in U_h$ defined on $[0, T]$, satisfying the following error estimate for $0 \leq t \leq T$:*

$$\begin{aligned} & \|u(\cdot, t) - u_h(\cdot, t)\|_1 \\ & \leq C_5 \left[\|u_0 - u_{0h}\|_1 + h^2 \left(|u_0|_3 + \int_0^t \left| \frac{\partial u(\cdot, s)}{\partial s} \right|_3 ds \right) \right]. \end{aligned} \quad (8.6.8)$$

8.6.2 Fully-discrete generalized difference schemes

Take the time step size $\tau = T/M$ and $t_k = k\tau$ ($k = 0, 1, \dots, M$). A fully-discrete generalized difference scheme for (8.6.1) is: Find $u_h^k \in U_h$ such that

$$\begin{cases} a(\bar{\partial}_t u_h^k, v_h) = \frac{1}{2} \left(\frac{\partial f(u_h^k)}{\partial x} + \frac{\partial f(u_h^{k-1})}{\partial x}, v_h \right), & v_h \in V_h, \\ k = 1, 2, \dots, M, \\ u_h^0 = u_{0h}, \end{cases} \quad (8.6.9a)$$

where $\bar{\partial}_t u_h^k = (u_h^k - u_h^{k-1})/\tau$. As before, we also consider the auxiliary problem

$$\begin{cases} a(\bar{\partial}_t u_h^k, v_h) = \frac{1}{2} \left(\frac{\partial \hat{f}(u_h^k)}{\partial x} + \frac{\partial \hat{f}(u_h^{k-1})}{\partial x}, v_h \right), & v_h \in V_h, \\ k = 1, 2, \dots, M, \\ u_h^0 = u_{0h}. \end{cases} \quad (8.6.10a)$$

$$(8.6.10b)$$

Lemma 8.6.3 For sufficiently small $\tau > 0$, problem (8.6.10) has a unique solution $\{u_h^k\} \subset U_h$, satisfying

$$\|u_h^k\|_1 \leq C_6 \|u_{0h}\|_1, \quad k = 1, 2, \dots, M. \quad (8.6.11)$$

Proof Suppose u_h^{k-1} is given. By the property (iii) of $a(u_h, \Pi_h^* v_h)$, for any $u_h \in U_h$ we have a unique $G u_h \in U_h$ such that

$$\begin{aligned} a(G u_h, \Pi_h^* w_h) &= a(u_h^{k-1}, \Pi_h^* w_h) + \frac{\tau}{2} \left(\frac{\partial \hat{f}(u_h)}{\partial x} + \frac{\partial \hat{f}(u_h^{k-1})}{\partial x}, \Pi_h^* w_h \right), \\ &\quad \forall w_h \in V_h. \end{aligned} \quad (8.6.12)$$

If we choose $w_h = G u_h$, then we have by (i)-(iii) that

$$\begin{aligned} \|G u_h\|_1 &\leq \frac{M}{\alpha} \|u_h^{k-1}\|_1 + \frac{\tau}{2} d''_\epsilon (\|u_h\|_1 + \|u_h^{k-1}\|_1) \\ &= \left(\frac{M}{\alpha} + \frac{\tau}{2} d''_\epsilon \right) \|u_h^{k-1}\|_1 + \frac{\tau}{2} d''_\epsilon \|u_h\|_1. \end{aligned}$$

Let $\tau > 0$ be sufficiently small such that $(1 - \frac{\tau}{2} d''_\epsilon) > 0$, and let R be a constant satisfying $R > (1 - \frac{\tau}{2} d''_\epsilon)^{-1} (\frac{M}{\alpha} + \frac{\tau}{2} d''_\epsilon) \|u_h^{k-1}\|_1$. Then $\|G u_h\|_1 \leq R$ when $\|u_h\|_1 < R$. But G is a continuous mapping. So by Brouwer fixed point theorem, G has a fixed point $u_h^k \in U_h$, which is exactly the solution to (8.6.10).

(8.6.10) implies that

$$a(u_h^k, v_h) = a(u_h^0, v_h) + \frac{\tau}{2} \sum_{l=1}^k \left(\frac{\partial \hat{f}(u_h^l)}{\partial x} + \frac{\partial \hat{f}(u_h^{l-1})}{\partial x}, v_h \right), \quad \forall v_h \in V_h.$$

Taking $v_h = \Pi_h^* u_h^k$ yields

$$\|u_h^k\|_1 \leq \frac{M}{\alpha} \|u_{0h}\|_1 + \frac{\tau}{2} d''_\epsilon \sum_{l=1}^k (\|u_h^l\|_1 + \|u_h^{l-1}\|_1),$$

and consequently

$$\|u_h^k\|_1 \leq (1 - \frac{\tau}{2} d''_\epsilon)^{-1} \left[\left(\frac{M}{\alpha} + \frac{\tau}{2} d''_\epsilon \right) \|u_{0h}\|_1 + d''_\epsilon \tau \sum_{l=1}^{k-1} \|u_h^l\|_1 \right].$$

So the discrete Gronwall's inequality guarantees the existence of a constant $C_8 > 0$ which validates (8.6.11).

Finally, we deal with the uniqueness of the solution. Let u_h^k and $\hat{u}_h^k$ be two solutions of (8.6.10). Then

$$\begin{aligned} & a(u_h^k - \hat{u}_h^k, v_h) \\ &= a(u_h^{k-1} - \hat{u}_h^{k-1}, v_h) + \frac{\tau}{2} \left(\frac{\partial \hat{f}(u_h^k)}{\partial x} - \frac{\partial \hat{f}(\hat{u}_h^k)}{\partial x}, v_h \right) \\ & \quad + \frac{\tau}{2} \left(\frac{\partial \hat{f}(u_h^{k-1})}{\partial x} - \frac{\partial \hat{f}(\hat{u}_h^{k-1})}{\partial x}, v_h \right), \quad \forall v_h \in V_h. \end{aligned}$$

Take $v_h = \Pi_h^*(u_h^k - \hat{u}_h^k)$ and note the boundedness of the solution, then we have

$$\begin{aligned} \|u_h^k - \hat{u}_h^k\|_1 &\leq \frac{M}{\alpha} \|u_h^{k-1} - \hat{u}_h^{k-1}\|_1 \\ &\quad + C\tau (\|u_h^k - \hat{u}_h^k\|_1 + \|u_h^{k-1} - \hat{u}_h^{k-1}\|_1). \end{aligned}$$

So for sufficiently small $\tau > 0$

$$\|u_h^k - \hat{u}_h^k\|_1 \leq C \|u_h^{k-1} - \hat{u}_h^{k-1}\|_1,$$

where $C > 0$ is a general constant. This recursive relation implies $u_h^k = \hat{u}_h^k$. This completes the proof. $\square$

Having done these preparations, we can prove the following theorem as in §5.2 and Theorem 8.6.1.

Theorem 8.6.2 *Assume that the solution to (8.6.1) is sufficiently smooth. Then for small enough h and τ , the fully-discrete scheme (8.6.9) has a unique solution $\{u_h^k\}$, and the following error estimate holds:*

$$\begin{aligned} & \|u_h^k(\cdot) - u(\cdot, t_k)\|_1 \\ &\leq C \left[\|u_0 - u_{0h}\|_1 + h^2 \left(|u_0|_3 + \int_0^{t_k} \left\| \frac{\partial u(\cdot, t)}{\partial t} \right\|_3 dt \right) \right. \\ & \quad \left. + \tau^2 \int_0^{t_k} \left\| \frac{\partial^2 u(\cdot, t)}{\partial t^2} \right\|_1 dt \right], \quad k = 1, 2, \dots, M. \end{aligned} \quad (8.6.13)$$

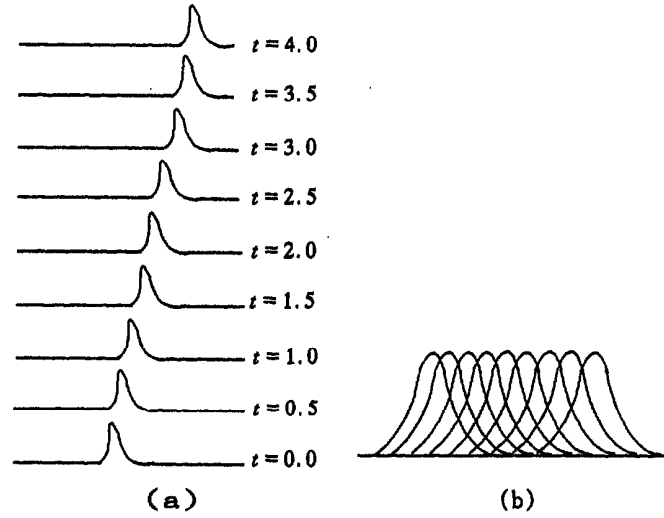


Fig. 8.6.1 Propagation of isolated waves

8.6.3 Numerical experiments

We use the generalized difference scheme (8.6.9) to approximate the regularized long wave equation (8.6.1), with $\alpha = \beta = 1$, $\gamma = 0.1$. In particular, we investigate the propagation of single isolated waves and the collision of double isolated waves, so as to check the efficiency of the scheme.

Propagation of single isolated waves

Set $a = 0$ and $b = 20$ in (8.6.1) and define the initial function as

$$u_0(x) = 3(c_0 - 1)(\operatorname{sech} X_0)^2, \quad X_0 = \sqrt{\frac{c_0 - 1}{4\gamma c_0}}(x - d_0),$$

where $c_0 = 2$, $d_0 = 8$. Choose the step sizes $h = 0.1$ and $\tau = 0.05$ in Scheme (8.6.9). The numerical results are depicted in Fig. 8.6.1(a). To investigate the propagation of the waves, all the waves are depicted simultaneously in Fig. 8.6.1(b). We observe from the figures that the isolated wave propagates forward with velocity $c_0 = 2$ and amplitude

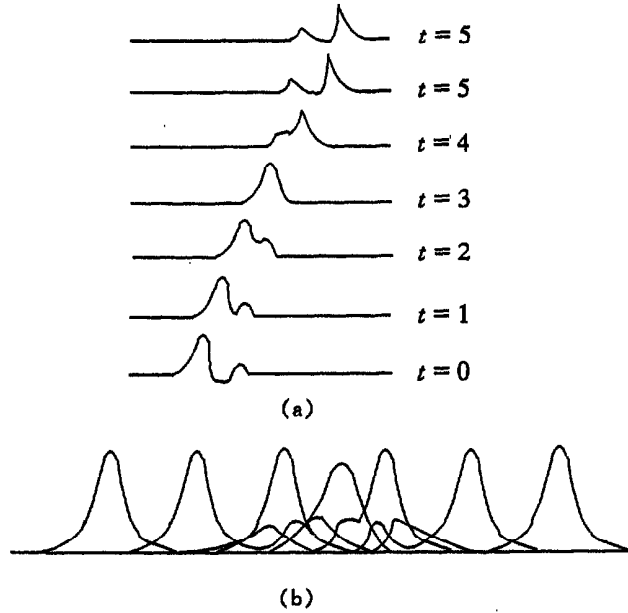


Fig. 8.6.2 Collision of double isolated waves

$3(c_0 - 1) \approx 3$, the wave form keeping unchanged. The numerical result matches quite well the theoretical isolated wave solution:

$$u(x, t) = 3(c_0 - 1)\text{sech}^2 X, \quad X = \sqrt{\frac{c_0 - 1}{4\gamma c_0}}(x - c_0 t - d_0).$$

Collision of double isolated waves

Choose $a = 0$, $b = 30$, $h = 0.1$, $\tau = 0.05$, and the initial function

$$u_0(x) = \sum_{i=1}^2 3(c_i - 1)(\text{sech} X_i)^2, \quad X_i = \sqrt{\frac{c_i - 1}{4\gamma c_i}}(x - d_i),$$

where $c_1 = 3$, $d_1 = 6$, $c_2 = 1.5$, $d_2 = 11.5$. The numerical results are shown in Fig. 8.6.2(a),(b). Amplify it twenty times in the longitudinal direction and truncate the heads of the waves to get Fig. 8.6.3.

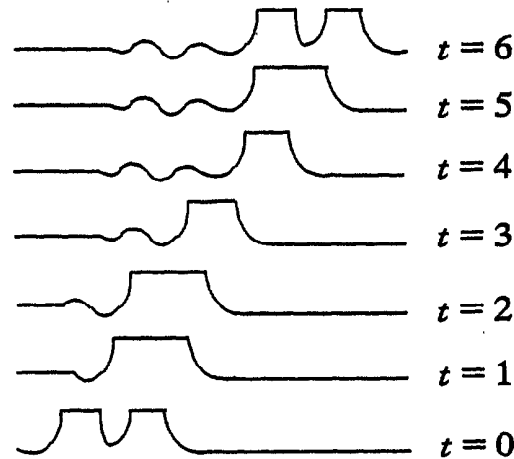


Fig. 8.6.3

One observes from this that before the collision, the big wave has a velocity $c_1 = 3$ and an amplitude $3(c_1 - 1) = 6$; and the small wave has a velocity $c_2 = 1.5$ and an amplitude $3(c_2 - 1) = 1.5$. During collision, the big wave gradually gets lower, while the small wave gets higher. After the collision, there appear once again a big wave and a small wave with almost identical amplitude and velocity as before the collision. But now apparently, the position of the big wave is (about 0.6) ahead of the position it would be at according to the original speed c_1 , while the position of the small wave is (about 1.1) behind according to the speed c_2 . We also notice the appearance of tail waves of slight vibrations.

8.7 Hierarchical Basis Methods

By now, we have established a theory for generalized difference methods, almost parallel to that for finite element methods. Generalized difference methods differs significantly from classical difference methods since they possess a variational form or a generalized Galerkin form. This advantage not only helps to establish the theory for generalized difference methods by use of finite element techniques, but also brings in the possibility to extend some algorithms designed for

finite element methods to finite difference methods. As an example, we discuss in this section the hierarchical basis methods for finite difference equations. (cf. [A-29] and [B-56].)

It is well-known that the condition number of the coefficient matrix of the discrete equation is $O(h^{-2})$ ($h > 0$ being the maximum step size of the grid), when we use the usual finite element or finite difference methods to solve a self-adjoint, positive definite, second order elliptic, planar, boundary value problem. Great progress was made in [B-100] by introducing a hierarchical basis for finite elements, resulting in a condition number $O((\log \frac{1}{h})^2)$. In the sequel, we shall first write the difference equation into a generalized difference form, then we shall make use of the hierarchical basis techniques to improve the condition number in like manner.

8.7.1 Hierarchical Basis

Let Ω be a polygonal region with boundary $\Gamma = \partial\Omega$, and let $\bar{\Omega} = \Omega \cup \Gamma$. Suppose T_0 is an initial triangulation of Ω . Starting from T_0 , we construct successively a series of triangulations of Ω : $T_0, T_1, \dots$. Each T_{k+1} is a uniform re-decomposition of the previous T_k , meaning that T_{k+1} is obtained by dividing each triangle of T_k into four equal smaller triangles with the three original vertexes and the three midpoints of the sides as new vertexes (cf. Fig. 8.7.1).

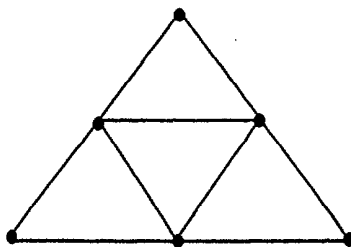


Fig. 8.7.1

Denote by N_k the set of all the nodes, i.e., the vertexes of the triangular elements of T_k . F_k denotes the piecewise linear, continuous function space relative to T_k . Call $u \in F_k$ a k -th hierarchy finite element function. For any $u \in C(\bar{\Omega})$, we use $I_k u \in F_k$ for the

interpolation projection of u onto F_k , that is,

$$I_k u \in F_k, (I_k u)(x) = u(x), \text{ when } x \in N_k. \quad (8.7.1)$$

Then, any $u \in F_j$ has the following decomposition

$$u = I_0 u + \sum_{k=1}^j (I_k u - I_{k-1} u). \quad (8.7.2)$$

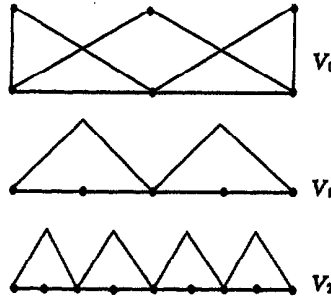


Fig. 8.7.2

Here each term is a rapidly oscillating function related to different hierarchies. $I_0 u$ is the finite element function relative to the initial grid T_0 . The function $(I_k u - I_{k-1} u) \in F_k$ vanishes on N_{k-1} . Set $V_k = \{u \in F_k; u(x) = 0 \text{ when } x \in N_{k-1}\}$. Apparently V_k is the range of $I_k - I_{k-1}$, and (8.7.2) means that F_j is a direct sum of $V_0 = F_0, V_1, \dots, V_j$. A standard finite element method takes the nodal basis functions of F_j as a basis. Here we introduce a hierarchical basis of F_j , consisting of the nodal basis functions of $V_0 = F_0, V_1, \dots, V_j$ (the corresponding nodes are respectively $N_0, N_1 \setminus N_0, \dots, N_k \setminus N_{k-1}, \dots, N_j \setminus N_{j-1}$). Some one-dimensional hierarchical basis functions are depicted in Fig. 8.7.2.

With respect to the decomposition (8.7.2), let us introduce a seminorm for $u \in F_j$

$$|u|^2 = \sum_{k=1}^j \sum_{x \in N_k \setminus N_{k-1}} |(I_k u - I_{k-1} u)(x)|^2. \quad (8.7.3)$$

Actually, $|u|$ is the Euclidian norm of the vector of the coefficients in the hierarchical basis expansion of u relative to F_j , of all the hierarchies except the initial one. Suppose the dimension of V_k is m_k ($k = 0, 1, \dots, j$), and the nodal basis functions are ϕ_{ki} ($i = 1, 2, \dots, m_k$). Then any $u \in F_j$ has the expression

$$u = \sum_{i=1}^{m_0} v_{0i} \phi_{0i} + \sum_{k=1}^j \sum_{i=1}^{m_k} v_{ki} \phi_{ki}, \quad (8.7.4)$$

where

$$\begin{cases} v_{0i} = u(x_i), & x_i \in N_0, \\ v_{ki} = (I_k u - I_{k-1} u)(x_{ki}), & x_{ki} \in N_k \setminus N_{k-1}. \end{cases} \quad (8.7.5)$$

If we unify the symbols ϕ_{ki} 's as $\phi_1, \dots, \phi_{m_0}, \phi_{m_0+1}, \dots, \phi_{m_0+m_1}, \dots, \phi_m$, where $m = \sum_{k=0}^j m_k$ is the dimension of F_j , then

$$u = \sum_{i=1}^{m_0} v_i \phi_i + \sum_{i=m_0+1}^m v_i \phi_i. \quad (8.7.6)$$

So it is clear that

$$|u|^2 = \sum_{i=m_0+1}^m |v_i|^2. \quad (8.7.3)'$$

Denote by $\|\cdot\|_0$ and $\|\cdot\|_1$ the norms of the Sobolev spaces $H^0 = L^2$ and H^1 respectively. The following important result is given in [B-100].

Theorem 8.7.1 *There exist constants $K_1^*, K_2^* > 0$, dependent on the diameter of Ω and the lower bound of the inner angles of the triangular elements but independent of the hierarchy number j , such that the following inequality holds for all $u \in F_j$:*

$$K_1^* (j+1)^{-2} (\|I_0 u\|_1^2 + |u|^2) \leq \|u\|_1^2 \leq K_2^* (\|I_0 u\|_1^2 + |u|^2). \quad (8.7.7)$$

8.7.2 Application to difference equations

Take the grid $T_h = T_j$ and the trial function space $U_h = F_j$. Choose a dual grid T_h^* of T_h , and the corresponding test function space V_h as the piecewise constant function space related to T_h^* . Let $a(u, v)$ be a bilinear form associated with a symmetric and positive definite second order elliptic operator on $\Omega \subset R^2$. The generalized difference method then reads: Find $u_h \in U_h$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \quad (8.7.8)$$

In particular, taking T_h as a rectangular grid results in a classical difference equation. Denote by Π_h^* the interpolation projection operator from U_h onto V_h . Then $a(u_h, \Pi_h^* \bar{u}_h)$ ($\bar{u}_h, u_h \in U_h$) is symmetric, and there exist positive constants γ_0 and γ_1 such that

$$\gamma_0 \|u_h\|_1^2 \leq a(u_h, \Pi_h^* u_h) \leq \gamma_1 \|u_h\|_1^2. \quad (8.7.9)$$

Inspired by this observation, we introduce for $u \in F_j$ the following norms:

$$|||u|||_1^2 = a(u_h, \Pi_h^* u_h); \quad |||u|||_0^2 = \|I_0 u\|_0^2 + |u|^2. \quad (8.7.10)$$

Then (8.7.9) can be written as

$$C_1 \|u\|_1^2 \geq |||u|||_1^2 \geq C_2 \|u\|_1^2, \quad \forall u \in F_j. \quad (8.7.11)$$

This together with Theorem 8.7.1 leads to

Theorem 8.7.2 *There exist constants K_1 and K_2 independent of the hierarchy number j such that*

$$K_1(j+1)^{-2} |||u|||_0^2 \leq |||u|||_1^2 \leq K_2 |||u|||_0^2, \quad \forall u \in F_j. \quad (8.7.12)$$

The constants K_1 and K_2 here depend on the lower bound of the inner angles of all the triangular elements, the diameter of Ω and the constants C_1 and C_2 . But they are independent of the regularity of the boundary value problems.

The coefficient matrix (the stiff matrix) of the generalized difference equation (8.7.8) has the following two expressions according to different bases of U_h and V_h .

Representation by nodal basis

Let $u_h = \sum_i u_i \phi_{x_i}$, where ϕ_{x_i} is the nodal basis of $x_i \in N_j$ and $u_i = u_h(x_i)$. We rewrite (8.7.8) in the form

$$\sum_i a(\phi_{x_i}, \Pi_h^* \phi_{x_l}) u_i = (f, \Pi_h^* \phi_{x_l}) = \hat{b}_l, \quad \forall x_l \in N_j. \quad (8.7.13)$$

Set $u = (u_1, u_2, \dots, u_m)^T$, $\hat{b} = (\hat{b}_1, \hat{b}_2, \dots, \hat{b}_m)^T$, and (a matrix) $\hat{A} = [a(\phi_{x_i}, \Pi_h^* \phi_{x_l})]_{m \times m}$. Then, (8.7.13) can be written as a vector form: $\hat{A}u = \hat{b}$. This gives a usual (nodal basis) generalized difference equation.

Representation by hierarchical basis

The difference solution u_h can be expressed as (8.7.6) by hierarchical bases. Accordingly we can write (8.7.8) as

$$\begin{aligned} & \sum_{i=1}^{m_0} a(\phi_i, \Pi_h^* \phi_l) v_i + \sum_{i=m_0+1}^m a(\phi_i, \Pi_h^* \phi_l) v_i \\ &= (f, \Pi_h^* \phi_l) = b_l, \quad l = 1, 2, \dots, m. \end{aligned} \quad (8.7.14)$$

Write $v = (v_1, \dots, v_{m_0}, v_{m_0+1}, \dots, v_m)^T$, $b = (b_1, b_2, \dots, b_m)^T$, and the matrix $A = [a(\phi_i, \Pi_h^* \phi_l)]_{m \times m}$. Then (8.7.14) becomes $Av = b$, called a hierarchical basis difference equation. The matrix A is obviously symmetric and positive definite.

Let B_0 be the m_0 order coefficient matrix on the upper left of the system (8.7.14), and

$$A_0 = \begin{pmatrix} B_0 & 0 \\ 0 & I \end{pmatrix}, \quad (8.7.15)$$

where I is the $(m - m_0) \times (m - m_0)$ identity matrix. It is clear that the norm $||| \cdot |||_0$ in (8.7.10) has the following representation

$$|||v|||_0^2 = \|I_0 v\|_0^2 + |v|^2 = \langle v, A_0 v \rangle.$$

Here $\langle \cdot, \cdot \rangle$ stands for the Euclidian inner product. Now Theorem 8.7.2 can be stated as: For any m -dimensional hierarchical basis coefficient

vector x , there holds (cf. (8.7.10))

$$K_1(j+1)^{-2}\langle x, A_0x \rangle \leq \langle x, Ax \rangle \leq K_2\langle x, A_0x \rangle. \quad (8.7.16)$$

Since the order m_0 of B_0 is not large, it is economical to compute the Cholesky decomposition $A_0 = LL^T$. Then we insert this decomposition into (8.7.16) to obtain

$$\begin{aligned} & K_1(j+1)^{-2}\langle L^Tx, L^Tx \rangle \\ & \leq \langle L^Tx, L^{-1}A(L^{-1})^TL^Tx \rangle \leq K_2\langle L^Tx, L^Tx \rangle. \end{aligned}$$

Thus we have the following estimate for the spectral condition number

$$\text{cond}(L^{-1}A(L^{-1})^T) \leq \frac{K_2}{K_1}(j+1)^2. \quad (8.7.17)$$

Here the factors L^{-1} and $(L^{-1})^T$ are not very important. They are mainly used to eliminate the geometrical influence of the initial triangulation. Since the low dimensional initial space F_0 is fixed and independent of the number of the hierarchies, the spectral condition number of LL^T is independent of number of the hierarchies, too. Furthermore, we note

$$\frac{\langle Ax, x \rangle}{\langle x, x \rangle} = \frac{\langle L^Tx, L^{-1}A(L^{-1})^TL^Tx \rangle}{\langle L^Tx, L^Tx \rangle} \frac{\langle L^Tx, L^Tx \rangle}{\langle x, x \rangle}.$$

Therefore, the condition number of the matrix A amplifies only in an order $O(j^2)$. Notice $j \sim \log \frac{1}{h}$. Hence, as in the case of finite element methods, the spectral condition number is reduced to $O\left(\left(\log \frac{1}{h}\right)^2\right)$.

Unfortunately A is no longer a sparse matrix, but it bears a simple relationship with the nodal basis coefficient matrix $\hat{A}$ (cf. (8.7.13)). Let S be such a transformation matrix which transforms the functions in F_j from a hierarchical basis representation into a nodal basis representation. So for any m -dimensional vectors $\bar{x}$ and $\bar{y}$ we have

$$\langle \bar{x}, A\bar{y} \rangle = \langle S\bar{x}, \hat{A}S\bar{y} \rangle = \langle \bar{x}, S^T \hat{A} S \bar{y} \rangle, \quad (8.7.18)$$

which implies

$$A = S^T \hat{A} S. \quad (8.7.19)$$

Our difference system is

$$\hat{A}x = \hat{b}. \quad (8.7.20)$$

where x is the difference solution vector. This system is equivalent to

$$Ay = S^T \hat{b}, \quad (\text{or resp. } L^{-1}A(L^{-1})^T y = L^{-1}S^T \hat{b},) \quad (8.7.21)$$

where $A = S^T \hat{A}S$ is the hierarchical basis coefficient matrix, and the nodal basis solution vector

$$x = Sy. \quad (\text{resp. } x = S(L^{-1})^T y.) \quad (8.7.22)$$

Since the condition number of A (resp. $L^{-1}A(L^{-1})^T$) is comparatively small, the convergence rate of an iteration method for (8.7.21) will be remarkably increased.

8.7.3 Iteration methods

As we see from (8.7.19), (8.7.21) and (8.7.22), it is a key point how to evaluate S . As a transformation matrix from a hierarchical basis representation into a nodal basis representation, S can be decomposed into $S = S_j S_{j-1} \cdots S_1$, where S_k describe the computation of the values of the new nodes on k -th hierarchy in terms of the values of the nodes on $(k-1)$ -th hierarchy. The diagonal entries of S_k are equal to 1; In the i -th row of S_k , there are two off-diagonal entries $\frac{1}{2}$ for each $x_i \in N_k \setminus N_{k-1}$; All the other entries are zero. So we can choose some fast algorithms to evaluate $S_k x$ and $S_k^T x$, with only $O(m)$ operations of multiplications and divisions ([cf. [B-100]]). If we choose such iteration methods that involve only operations like Ax , then we can avoid the difficulties such as the complex structure and the increased nonzero entries of A . The following two examples illustrate the idea.

Richardson iteration method

Applying the Richardson method to (8.7.21) yields

$$y^{k+1} = y^k - \omega(S^T \hat{A}S y^k - S^T \hat{b}), \quad k = 0, 1, \dots \quad (8.7.23)$$

If we choose the optimal relaxation factor $\omega_{\text{opt}} = 2/(\alpha + \beta)$, where α and β are the maximum and minimum eigenvalues respectively of the

symmetric and positive definite matrix A , then the convergence rate of (8.7.23) is $O(j^{-2})$, and the computational amount of each step is only $O(m)$.

Conjugate gradient method

The convergence rate of this method for (8.7.21) is $O(j^{-1})$, and the computational amount of each step is also $O(m)$.

Block Gauss-Seidel method and successive block over-relaxation method can as well be used for (8.7.21). (See [A-29].)

8.7.4 Numerical experiments

Consider the first boundary value problem of The Laplacian equation:

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0, & \text{on } \Omega = (0, 1) \times (0, 1), \\ u|_{\partial\Omega} = 0. \end{cases}$$

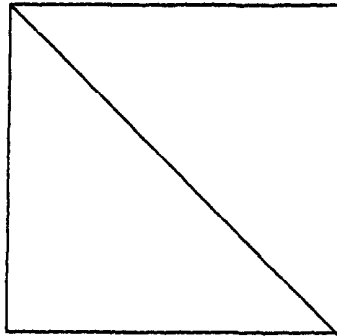


Fig. 8.7.3

Its unique solution is $u = 0$. Based on the initial grid in Fig. 8.7.3, we place a series of densifying triangulations. In this case, the nodal difference equation is precisely the well-known five point difference scheme. The following methods are used: nodal basis Gauss-Seidel method (N-GS in short), hierarchical basis Richardson method (H-Richardson), conjugate gradient method (H-CG), block Gauss-Seidel method (H-BGS) and block successive over-relaxation method (H-BSOR). The numbers of iteration steps to reach the accuracy 10^{-3} are shown in the following table. We observe that the convergence

rates of the hierarchical methods (H-) are significantly higher than that of the usual iteration method (N-GS).

method	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
N-GS	11	47	191	767	3067
H-Richardson	14	32	58	92	134
H-CG	4	11	16	22	28
H-BGS	9	13	19	26	35
H-BSOR	7	9	10	12	16

Bibliography and Comments

The paper [B-65] by R.H. MacNeal (1953) is the earliest work studying the difference methods on irregular networks, which was originally proposed to simulate an electric network. A.M. Winslow [B-99] (1967) extended this method to the computation of electromagnetic fields. Further extensions and applications were discussed in [A-60] and [B-72]. A direct extension to three-dimensional regions is presented in this chapter (§8.2).

Early in the seventies, [A-14,49] studied the difference methods on triangular grids for elastic problems, which were called discretization operator methods.

Up to now, fluid mechanics, especially underground fluid mechanics, may be the field where the generalized difference methods apply most often and most successfully, represented by the works [A-38,39,40,13] and [B-29,38]. An important feature of generalized difference methods is that they keep up the mass conservation. Perhaps this is the reason why computational fluid researchers are attracted to them. The application of generalized difference methods in aeromechanics is also fairly successful, cf §6.4 and the corresponding references in the end of the book.

[A-48] proposes a significant extension of the staggered scheme (the nodes of the pressure and the velocity being distributed alternately) for incompressible flows. [A-56] discussed a coupled sound-

heat problem, but only in one dimension. [A-4] studied a regularized long wave problem, which was a nonlinear problem, and obtained satisfactory wave forms by a quadratic element generalized difference scheme.

[A-29] and [B-56] extended the hierarchical basis method for finite element methods to finite difference methods, and highlighted an approach to transplant an algorithm for finite element methods to one for finite difference methods. Finite difference methods were used in [A-33] to compute the long time behaviour of dynamical systems, which provided another example of a successful extension of finite element theories to finite difference methods.

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Index

- Affine mapping, 17
- Area coordinates, 11, 116
- Averaging function, 2
- Bilinear forms, 20, 37
 - boundedness, 27, 28, 37
 - coercive, 22
 - positive definite, 22, 28, 37, 40, 42
 - symmetric, 22
 - V-elliptic, 22
- Boundary conditions
 - essential, 47
 - natural, 47
- Characteristic direction, 326
- Coercive condition, 22
- Collocation methods, 31
- Continuous extension, 21, 35, 39, 41
- Continuous function spaces
 - $C^m([a, b]; X)$, 275
 - $C^m(\Omega)$, 1
 - $C_0^m(\Omega)$, 1
- Continuous imbedding, 20
- C-uniform, 179
- Decomposition, 10, 114, 131
 - quadrilateral, 131
 - triangular, 10, 114
- Differential equations
 - biharmonic, 26, 187
 - convection-diffusion, 326, 344
 - coupled sound-heat, 394
 - electromagnetic field, 367
 - first order hyperbolic, 296
 - hyperbolic-parabolic, 394
 - nonlinear elliptic, 187, 227, 356
 - nonlinear parabolic, 235, 272
 - parabolic, 235
 - planar elastic, 361
 - Poisson, 25
 - regularized long wave, 399
 - second order elliptic, 111
 - second order hyperbolic, 287
 - second order ODE, 45
 - Stokes, 385
 - system of first order hyperbolic, 307
 - three-dimensional Poisson, 371
 - water pollution, 373
- Downwind value, 299
- Dual argument, 37
- Dual decompositions (grids), 52, 55, 63, 75, 115, 132, 140, 154, 369, 370, 387

- barycenter, 115
 - central, 132
 - circumcenter, 115
- Dual operator, 32
- Dual space, 32
- Elliptic projection, 240
- Error estimates, 33, 37, 40, 42
 - in H^0 norm, 88, 90, 167, 303, 304, 350
 - in H^1 norm, 58, 69, 86, 129, 137, 150, 164, 233, 292, 296, 338
 - in H^2 norm, 106, 214, 226
 - in $W^{0,\infty}$ norm, 91, 173
- Extreme value theorem, 338, 342
- Finite elements
 - Adini, 220
 - Adini rectangular, 15
 - Bogner-Fox-Schmidt, 15
 - Hermite cubic, 12
 - inverse property of, 19
 - Lagrange bilinear, 13
 - Lagrange bi-quadratic, 14
 - Lagrange linear, 11
 - Lagrange quadratic, 11
 - Zienkiewicz, 12, 202, 204, 205
- Finite element space, 9, 10
- Finite volume method, 317
- Fundamental lemma of variational methods, 3
- Galerkin generalized solutions, 21
- Galerkin methods, 30, 31, 35, 47, 48
- Generalized difference methods, 52, 76, 101, 114, 131
 - backward Euler, 245
 - conservative, 120
 - Crank-Nicolson, 245
 - cubic element, 72, 154, 261
 - generalized upwind, 296, 309, 332, 344, 356, 378
 - highly accurate generalized upwind, 350
 - linear element, 53, 119
 - mass concentration, 235, 254
 - mixed, 190, 197
 - nonconforming, 202, 220, 386
 - quadratic element, 61, 139
- Generalized Galerkin methods, 38, 43
- Godunov scheme, 317, 320
- Green's formulas, 23
- Hierarchical basis methods, 410
- Interpolation operator, 16
- Interpolation operator Π_h , 59, 69, 118
- Interpolation operator Π_h^* , 58, 67, 118, 157
- Interpolation weak estimates, 94, 96, 98, 174
- Isolated waves, 407
- Lax-Friedrichs scheme, 320
- Least square method, 31
- Line segment $\overline{PQ}$, 115
- L -region, 5
- Mass conservation, 343
- Monotone scheme, 323

Nonconforming method, 113

Norms

$C^m([a, b]; X)$, 275

discrete, 66, 79, 124, 145,
158

$L^p((a, b); X)$, 275

(m, p) , 4

$(-m, p)$, 7

quotient, 16

semi-, 5

Numerical flux, 320

Optimal stress points, 93

Osher's split technique of non-
linear functions, 356

Petrov-Galerkin method, 31

P_K -interpolation, 16

Positive definite condition, 22

Projection methods, 30

Projection operators, 30, 31, 36

Quadratic functional, 22

minimum of, 22

Quasi-parallelogram, 179

condition, 272

Quasi-uniform, 19, 54

Reference element, 17, 132

Regular family, 17

Riesz

generalized solution, 22

representation theorem, 20,
28, 36, 39, 41

variational principle, 23

Sobolev spaces, 1, 4

$H^m(\Omega)$, 4

$H_0^m(\Omega)$, 4

$H^{-m}(\Omega)$, 7

interpolation in, 15, 43

$L^p((a, b); X)$, 275

$L^p(\Omega)$, 1

$W^{-m,p'}(\Omega)$, 7

$W^{m,p}(\Omega)$, 4

$W_0^{m,p}(\Omega)$, 4

Stable, 85

Superconvergence, 92-94, 96, 97,
100, 174, 182

Theorems

average approximation, 2

Babuska, 28

equivalent norm, 5

imbedding, 7, 8

inverse property, 19

Lax-Milgram, 27

trace, 5

Trace, 5, 8

Trial and test spaces, 39, 41,

53, 61, 73, 114, 131,

139, 154

Uniquely approximate-solvable,
33

Upwind value, 299

Variational principles, 22, 204

Ciarlet-Raviart, 187

generalized Galerkin, 49, 51

Hermann-Miyoshi, 196

potential energy, 23

Riesz, 23

virtual work, 22, 47

V-elliptic condition, 22

V_h -interpolation, 16

Volume coordinates, 368

Weak convergence, 32, 34

⋮